IV. *On the Induction of Electric Currents in Infinite Plates and Spherical Shells.*

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Introduction.

§ 1. In Vol. XX. (1872) of the Proceedings of the Royal Society (pp. 160–168) is a beautiful paper by the late Professor Clerk Maxwell giving an investigation of the induction of currents in an infinite plane sheet of uniform conductivity. For the purposes of the investigation the sheet is supposed infinitely thin; and when it is at rest and influenced by a varying external magnetic system, the effect of the currents induced in it is found to be equivalent to an infinite train of images, at the sheet, of the external system, which, after being formed, move off to infinity with uniform velocity. When the external system revolves uniformly round an axis normal to the sheet, the effect is shown to be the same as if the sheet itself revolved round the axis and the magnetic system remained fixed. The images will then lie in a spiral trail in the form of a helix whose axis is perpendicular to the sheet. This theory was afterwards reproduced in his 'Treatise on Electricity and Magnetism,' and the latter part proved directly from the equations. The analysis there given is somewhat difficult to follow, though it is doubtless possible to present it in a more logically exact form.

The problem of the induction of currents has also been treated by Felici (Tertolini's 'Annali,' 1853–54) and by Jochmann (Crelle, 1864, and Pogg. Ann., 1864). Jochmann has solved the case of a sphere which rotates uniformly in a magnetic field symmetrical about the axis of revolution and finds that no currents will be generated in it, but that there will be a certain distribution of free electricity throughout its interior and over its surface. He has also handled the case of an infinite plate of finite thickness, which revolves uniformly round a normal, by neglecting the inductive action of the currents on themselves, and shows that the conditions of the problem may then be satisfied by a system of currents parallel to the faces of the plate; he has also traced the forms of the current and equipotential lines in some simple cases. The solution, however, as Maxwell has shown in the case of a thin copper disc, can be true only for *very* small values of the angular velocity.

HELMHOLTZ, in an elaborate memoir on the "Equations of Motion of Electricity," in CRELLE'S Journal (vols. 72, 78), has given an exhaustive analysis of the conditions which have to be satisfied in any problem regarding the movement of electricity, and has proved very clearly that the solution of any problem is unique; but he has not dealt with any special case of the problem of induced currents.

MAXWELL'S investigation remains up to the present, so far as I am aware, the only case in which the complete solution of any case of induction has been published.

German writers on current electricity have usually adopted some form of the theory of action at a distance between the elements of different currents, and the free electricity is conceived as a scalar quantity distributed with a certain density throughout the interior and over the surface of conductors. MAXWELL'S theory, which is adopted in the present paper, though it leads generally to similar equations, differs notably from the other in both these respects. The energy is supposed to be seated everywhere in the surrounding medium, and the free electricity is the convergence of a vector quantity termed the electric displacement. The total current, to which electro-magnetic phenomena are due, is compounded of the current of conduction and the time-variations of the electric displacement. Owing to this peculiarity of the theory, the conditions to be satisfied at the surface of separation of two substances will differ from those given by HELMHOLTZ. I have therefore analysed them somewhat fully: taking first, for the sake of generality and the simplicity which it gives, the most general case of two substances in which both the conductivity and specific inductive capacity are to be retained. We can then deduce the conditions at the common surface of two conductors, or of a conductor and a dielectric, which is the case with which we have to do.

One special result of these conditions is that when the vector potential, at the surface, due to all the currents or magnets in the field is at each point perpendicular to the surface of the conductor, the electric potential will vanish everywhere, and there will be no free electricity present either in the conductor or on its surface.

This happens in the case of an infinite plane plate of any thickness. The vector potential (or electro-magnetic momentum) is then everywhere parallel to the surface of the plate, and is derived by vector differentiation from a function P; the current in it is also everywhere parallel to the surface and is derived from a single current function Φ. It also appears that P is the potential of imaginary matter distributed with density Φ; and, during the decay of the currents, P satisfies an equation of the same form as that which regulates the diffusion of heat throughout a solid.

When the plate is infinitely thick there will be no reaction in the inducing system; but when it is very thin, the effect will be that given by MAXWELL as already explained. The general formulæ in this case reproduce his results.

For a solid sphere or shell bounded by two concentric spherical surfaces, the vector potential and current are everywhere at right angles to the radius vector to the common centre, and their values may be derived from two functions, P and Φ, which are related to each other as in a plane plate; during its decay also, P follows the same law as in that case.

When the shell is infinitely thin, the effect, on an external point, of the currents excited in it may be represented by the following system of images, which constitute a generalisation of those of MAXWELL. Divide the time into an infinite number of equal intervals, and at the commencement of any of these let a positive image of the system be formed in the place occupied by its electric image at the surface. Let the parts of this image move towards the centre in straight lines so that the logarithmic decrement of their distances from the centre is constant and equal to $\frac{\mathrm{R}}{2\pi \mathrm{a}}$ (R being the resistance of the shell and a its radius), and let the intensity of the image increase at each point with a constant logarithmic rate $\frac{\mathrm{R}}{4\pi \mathrm{a}}$. At the end of the interval let an exactly equal but negative image be formed in the place of the former and move towards the centre in the same manner, and let these operations be repeated at the commencement and end of every interval during which the external system is varying; the action of the sheet on external points will be that due to the above train of images. The action on a point within the sheet may be represented in a somewhat similar manner.

When the shell possesses a finite thickness, or is a solid sphere, it is not possible to express its effect so simply. The variations in the external system produce continually new systems of currents, the law of whose decay may be exhibited by expressing P in a series of terms containing each the product of a tesseral harmonic, a "spherical" function of the radius, and an exponential $e^{-\lambda t}$, the coefficients of which are to be found by known methods.

When the shell degrades into an infinite plate, the "spherical" function becomes an exponential or circular function, and the tesseral harmonic becomes the product of a factor $\cos m\phi$ or $\sin m\phi$ by a BESSEL'S function $\mathrm{J}_m(\kappa\rho)$. The coefficients might then be found by means of NEUMANN'S theorem for expanding $f(x, y)$ in BESSEL'S functions; but their deduction from the corresponding problem of spherical harmonic analysis throws an interesting light upon NEUMANN'S expansion, and especially on the meaning of the symbol ∞ in the limits of integration.

When a symmetrical conductor revolves uniformly about its axis of symmetry for a sufficient length of time, the currents and electric distribution become steady, and the total currents are then identical with the currents of conduction. In the case of a plate or spherical shell, the vector potential and currents are expressible in the same manner as before in terms of two functions P and Φ, which are still related to each other as formerly. The equation which now determines P is $\frac{\sigma}{4\pi}\nabla^2\mathrm{P}=\omega\frac{d(\mathrm{P}+\mathrm{P}_0)}{d\phi}$.

The general results of calculation verify MAXWELL'S theorem of the spiral trail of images due to an infinitely thin plate. The theorem is also extended to a spherical current sheet; the trail here becomes conical, the locus of points in it, which correspond to a given point of the influencing system, forming a curve which would become portions of an equiangular spiral, if the cone on which it lies were developed on a plane.

The determination of the coefficients in the problems having reference to a sphere or spherical shell depends on the elementary formulæ of reduction of the spherical functions. By adopting a particular mode of constructing the theory of these expressions, it is possible to obtain the necessary properties almost immediately from the definitions: a short sketch of the subject is therefore given, confined, however, to the results which are necessary for the subsequent analysis.

General equations.

§ 2. The general equations of the field in MAXWELL'S theory are expressed in terms of

The electro-magnetic momentum at a point,	F, G, H
The magnetic induction	a, b, c
The magnetic force	α, β, γ
The total electric current	u, v, w
The current of conduction	p, q, r
The electric displacement	f, g, h
The electromotive force	P, Q, R
The velocity of a point	$\dot{x}, \dot{y}, \dot{z}$

We have also the following scalar quantities,

The electric potential ψ,
The magnetic potential Ω,
The conductivity for electric currents C,
The resistance to conduction, $\sigma=\frac{1}{C}$,
The dielectric inductive capacity K,
The free electricity at any point in the substance of the conductor ϵ per unit of volume, on the surface ϵ' per unit of area.

Since the dielectric surrounding the conductors is air, we may treat the magnetic induction and magnetic force as identical, and we may put

$$\alpha=\frac{dH}{dy}-\frac{dG}{dz}, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The total electric current u, v, w is connected with the magnetic force by equations of the form,

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz},\ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and satisfies the equation of continuity

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

If we put

$$u=p+\frac{df}{dt},\ .\ .\ .\ ,\ \epsilon=\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz},$$

$$\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}+\frac{d\epsilon}{dt}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

expressing that the loss of electricity by conduction through the faces of an element is equal to the loss of free electricity in the element, a result which may be taken as self evident.

If we put, with MAXWELL, $p=\text{CP}$, $f=\frac{\text{K}}{4\pi}\cdot\text{P}$, &c., this equation may be written

$$\text{C}\epsilon+\frac{\text{K}}{4\pi}\,\frac{d\epsilon}{dt}=0.$$

If ϵ_0 be the initial value of ϵ,

$$\epsilon=\epsilon_0 e^{-4\pi\frac{\text{C}}{\text{K}}t}$$

showing that any initial electrification will rapidly disappear in a conductor for which κ is small compared to C. When the substance does not conduct we shall have $\epsilon=\epsilon_0$; so that if we suppose air and other non-conductors initially uncharged, they cannot acquire any charge.

The equations which determine the vector potential or electro-magnetic momentum in terms of the current are

$$\text{F}=\frac{1}{4\pi}\int\frac{u'}{r}dx'dy'dz',\ \&c.,$$

where $r=\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, and the integrations are to be extended over all space where there are currents, whether these be currents due to conduction, or time-variations of the electric displacement. F, G, H are thus the potentials of distributions of imaginary matter of finite density; and, therefore, in crossing the surface which divides two substances in the field, we shall have

$$\left.\begin{array}{l} \mathrm{F}=\mathrm{F}', \quad \mathrm{G}=\mathrm{G}', \quad \mathrm{H}=\mathrm{H}' \\ \dfrac{d\mathrm{F}}{d\mathrm{N}}=\dfrac{d\mathrm{F}'}{d\mathrm{N}}, \ \dfrac{d\mathrm{G}}{d\mathrm{N}}=\dfrac{d\mathrm{G}'}{d\mathrm{N}}, \ \dfrac{d\mathrm{H}}{d\mathrm{N}}=\dfrac{d\mathrm{H}'}{d\mathrm{N}}, \end{array}\right\} \quad \dots\dots\dots \quad (4)$$

F′, G′, H′ being the components of the electro-magnetic momentum in the second substance, and $d\mathrm{N}$ an element of the normal drawn from the first into the second.

The other boundary conditions are more difficult to recognise. Let us first confine our attention to the common surface of separation of two substances at rest. In the first the equations of electromotive force are

$$\left.\begin{array}{l} \mathrm{P}=-\dfrac{d\mathrm{F}}{dt}-\dfrac{d\psi}{dx} \\ \mathrm{Q}=-\dfrac{d\mathrm{G}}{dt}-\dfrac{d\psi}{dy} \\ \mathrm{R}=-\dfrac{d\mathrm{H}}{dt}-\dfrac{d\psi}{dz} \end{array}\right\} \quad \dots\dots\dots\dots \quad (5)$$

with similar equations for the second. Let l, m, n be the direction cosines of the normal (N) at any point, and let $\mathfrak{F}=l\mathrm{F}+m\mathrm{G}+n\mathrm{H}$; then, since

$$u=\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\mathrm{P}, \text{ \&c.,}$$

$$lu+mv+nw=-\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\frac{d\mathfrak{F}}{dt}-\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\frac{d\psi}{d\mathrm{N}};$$

and, for the second substance,

$$lu'+mv'+nw'=-\left(\mathrm{C}'+\frac{\mathrm{K}'}{4\pi}\frac{d}{dt}\right)\frac{d\mathfrak{F}}{dt}-\left(\mathrm{C}'+\frac{\mathrm{K}'}{4\pi}\frac{d}{dt}\right)\frac{d\psi'}{d\mathrm{N}}.$$

The condition of "continuity" at the surface requires that

$$lu+mv+nw=lu'+mv'+nw',$$

and therefore

$$\left\{\mathrm{C}-\mathrm{C}'+\left(\frac{\mathrm{K}-\mathrm{K}'}{4\pi}\right)\frac{d}{dt}\right\}\frac{d\mathfrak{F}}{dt}+\left(\mathrm{C}+\frac{\mathrm{K}}{4\pi}\frac{d}{dt}\right)\frac{d\psi}{d\mathrm{N}}-\left(\mathrm{C}'+\frac{\mathrm{K}'}{4\pi}\frac{d}{dt}\right)\frac{d\psi'}{d\mathrm{N}}=0 \quad \dots \quad (6)$$

The free electricity ϵ' at the surface is given by

$$\begin{aligned} 4\pi\epsilon' &= \mathrm{K}\left(-\frac{d\mathfrak{F}}{dt}-\frac{d\psi}{d\mathrm{N}}\right)-\mathrm{K}'\left(-\frac{d\mathfrak{F}}{dt}-\frac{d\psi'}{d\mathrm{N}}\right) \\ &= (\mathrm{KC}'-\mathrm{K}'\mathrm{C})\left(\mathrm{C}-\mathrm{C}'+\left(\frac{\mathrm{K}-\mathrm{K}'}{4\pi}\right)\frac{d}{dt}\right)^{-1}\left(\frac{d\psi}{d\mathrm{N}}-\frac{d\psi'}{d\mathrm{N}}\right) \quad \dots\dots \quad (6,) \end{aligned}$$

(*a.*) When both the substances are good conductors, we may take $K=K'=0$; the equations then become

$$\left.\begin{aligned}(C-C')\frac{d\mathfrak{F}}{dt}+C\frac{d\psi}{dN}-C'\frac{d\psi'}{dN}=0\\ \epsilon'=0\end{aligned}\right\}\quad \ldots\ldots\ldots\ldots \quad (7)$$

(*b.*) When the first substance is a conductor and the second a non-conductor (air), we shall have $K=0$, $C'=0$; the equations are then, electro-magnetic measurements being still employed,

$$\left.\begin{aligned}\left(C-\frac{K'}{4\pi}\frac{d}{dt}\right)\frac{d\mathfrak{F}}{dt}+C\frac{d\psi}{dN}-\frac{K'}{4\pi}\frac{d}{dt}\cdot\frac{d\psi'}{dN}=0\\ 4\pi\epsilon'=-K'C\left(C-\frac{K'}{4\pi}\frac{d}{dt}\right)^{-1}\left(\frac{d\psi}{dN}-\frac{d\psi'}{dN}\right)\end{aligned}\right\}\quad \ldots\ldots \quad (8)$$

But K' is infinitely small compared to C; and therefore, if we write (ϵ') for the electrostatic measure of ϵ', we shall have

$$\left.\begin{aligned}\frac{d\mathfrak{F}}{dt}+\frac{d\psi}{dN}=0\\ 4\pi(\epsilon')=\frac{d\psi'}{dN}-\frac{d\psi}{dN}\end{aligned}\right\}\quad \ldots\ldots\ldots\ldots \quad (9)$$

We may derive an important corollary from these results.

If $\mathfrak{F}$ is always $=0$ at the surface of the conductors (as will be the case in the following problems), we shall have

$$\frac{d\psi}{dN}=0 \quad \ldots\ldots\ldots\ldots \quad (10)$$

But from equations (5), remembering that $\epsilon=0$, we derive

$$\nabla^2\psi=0,\ \text{where}\ \nabla^2=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2} \quad \ldots\ldots\ldots \quad (11)$$

within the conductor, therefore, ψ is everywhere zero; and since ψ' always satisfies the equation $\nabla^2\psi'=0$ outside the surface and is zero at every point of it, it follows that ψ' is also everywhere zero, and there is no free electricity either within or upon the conductor.

Let us now collect the results of the above discussion, so far as it relates to a conductor disturbed by the introduction or variation of a magnetic system in a space surrounded by air; putting $K=0$, and writing σ for $\frac{1}{C}$, (1.) Within the conductor

$$\frac{\sigma}{4\pi}\nabla^2\mathrm{F}=-\frac{d\mathrm{F}}{dt}-\frac{d\psi}{dx}$$

$$\frac{\sigma}{4\pi}\nabla^2\mathrm{G}=-\frac{d\mathrm{G}}{dt}-\frac{d\psi}{dy}$$

$$\frac{\sigma}{4\pi}\nabla^2\mathrm{H}=-\frac{d\mathrm{H}}{dt}-\frac{d\psi}{dz}$$

$$\epsilon=0,\ \nabla^2\psi=0.$$

(2.) Outside the conductor $\mathrm{C}'=0$, and K' may be neglected ; therefore

$$\nabla^2\mathrm{F}'=0,\ \nabla^2\mathrm{G}'=0,\ \nabla^2\mathrm{H}'=0,$$

$$\epsilon''=0,\ \nabla^2\psi'=0,$$

(ϵ''=electric density outside).

(3.) At the surface of the conductor

$$\mathrm{F}=\mathrm{F}',\ \frac{d\mathrm{F}}{d\mathrm{N}}=\frac{d\mathrm{F}'}{d\mathrm{N}},\ \&\text{c.}$$

$$\frac{d\mathfrak{F}}{dt}+\frac{d\psi}{d\mathrm{N}}=0,\ \mathfrak{F}=l\mathrm{F}+m\mathrm{G}+n\mathrm{H},\ \psi=\psi'$$

$$4\pi(\epsilon')=\frac{d\psi'}{d\mathrm{N}}-\frac{d\psi}{d\mathrm{N}}.$$

(4.) When $\mathfrak{F}=0$, $\psi=0$, $\psi'=0$, $\epsilon'=0$, and the currents are confined to the conductor.

When the conducting substance is moving in any manner, the equations of electromotive force are (MAXWELL, Vol. II., Art. 598)

$$\mathrm{P}=\gamma\dot{y}-\beta\dot{z}-\frac{d\mathrm{F}}{dt}-\frac{d\psi}{dx}$$

$$\mathrm{Q}=\alpha\dot{z}-\gamma\dot{x}-\frac{d\mathrm{G}}{dt}-\frac{d\psi}{dy}$$

$$\mathrm{R}=\beta\dot{x}-\alpha\dot{y}-\frac{d\mathrm{H}}{dt}-\frac{d\psi}{dz}$$

If we differentiate these with respect to x, y, z and add, putting for $\frac{d\mathrm{P}}{dx}+\frac{d\mathrm{Q}}{dy}+\frac{d\mathrm{R}}{dz}$ its electrostatic value $4\pi(\epsilon)$, we find

$$\begin{aligned}4\pi(\epsilon)&=\dot{x}\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)+\ \ldots\ +\alpha\left(\frac{d\dot{z}}{dy}-\frac{d\dot{y}}{dz}\right)+\ \ldots\ -\nabla^2\psi\\&=-\frac{1}{4\pi}(u\dot{x}+v\dot{y}+w\dot{z})+2(\alpha\dot{\omega}_1+\beta\dot{\omega}_2+\gamma\dot{\omega}_3)-\nabla^2\psi\ .\quad .\quad .\quad .\qquad (12)\end{aligned}$$

where $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$ are the angular velocities of the element at x, y, z. We cannot, therefore, look upon ψ as the potential of free electricity; in fact it is easy to see that there will be no free electricity inside it, just as when it was at rest. If the conductor be symmetrical about an axis, and revolve about it sufficiently long for the currents to become steady, the total current will become identical with the conduction current, and there will be no flow across the surface. We may then take at every point of the surface

$$lu+mv+nw=0.$$

The triple integrals in the expressions for F, G, H are then to be taken only throughout the conductor, and we shall have as before F and $\frac{d\mathrm{F}}{d\mathrm{N}}$, &c., continuous in passing across the surface; and outside, F, G, H satisfy the equations

$$\nabla^2\mathrm{F}=0,\quad \nabla^2\mathrm{G}=0,\quad \nabla^2\mathrm{H}=0.$$

The special problems which are solved in the present paper depend for their solution on certain properties of the vector potential; and it will therefore not be out of place to devote a little space to their preliminary discussion. We shall thereby gain a clearer insight into the subsequent analysis.

The vector potential.

§ 3. (A.) We shall first examine the nature of the vector potential inside a space due to magnets or currents outside that space. It is connected with the magnetic potential Ω by the equations.

(*a*.)

$$\left.\begin{aligned}\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz}&=-\frac{d\Omega}{dx}\\ \frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{dx}&=-\frac{d\Omega}{dy}\\ \frac{d\mathrm{G}}{dx}-\frac{d\mathrm{F}}{dy}&=-\frac{d\Omega}{dz}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (13)$$

and it is clear that, if we can find one set of values F, G, H satisfying the equations when Ω is supposed given, the complete values will be

$$\mathrm{F}+\frac{d\chi}{dx},\ \mathrm{G}+\frac{d\chi}{dy},\ \mathrm{H}+\frac{d\chi}{dz},$$

where χ is an arbitrary function of x, y, z. It is therefore necessary for our purpose to find one solution only.

If we can express Ω in the form

$$\Omega=\left(\mathrm{A}\frac{d}{dx}+\mathrm{B}\frac{d}{dy}+\mathrm{C}\frac{d}{dz}\right)\mathrm{P}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

where P satisfies the equation $\nabla^2\mathrm{P}=0$, the equations are satisfied by

$$\left.\begin{aligned}\mathrm{F}&=\left(\mathrm{B}\frac{d}{dz}-\mathrm{C}\frac{d}{dy}\right)\mathrm{P}\\ \mathrm{G}&=\left(\mathrm{C}\frac{d}{dx}-\mathrm{A}\frac{d}{dz}\right)\mathrm{P}\\ \mathrm{H}&=\left(\mathrm{A}\frac{d}{dy}-\mathrm{B}\frac{d}{dx}\right)\mathrm{P}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

and, in particular, if

$$\left.\begin{aligned}\Omega&=\frac{d\mathrm{P}}{dz}\\ \mathrm{F}&=-\frac{d\mathrm{P}}{dy},\ \mathrm{G}=\frac{d\mathrm{P}}{dx},\ \mathrm{H}=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

These expressions (15) may be easily verified by actual differentiation, taking account of $\nabla^2\mathrm{P}=0$, and may be looked upon as a generalisation of the equations given by MAXWELL ('Electricity,' Art. 405). They give additional interest to his expression for a solid negative harmonic

$$\left(\mathrm{A}_1\frac{d}{dx}+\mathrm{B}_1\frac{d}{dy}+\mathrm{C}_1\frac{d}{dz}\right)\ldots\left(\mathrm{A}_n\frac{d}{dx}+\mathrm{B}_n\frac{d}{dy}+\mathrm{C}_n\frac{d}{dz}\right).\frac{1}{r};$$

and in forming F, G, H any one of the n factors may be chosen to furnish A, B, C; the results obtained by taking two different factors differing by quantities of the form $\frac{d\chi}{dx}, \frac{d\chi}{dy}, \frac{d\chi}{dz}$, as I have verified. When Ω is a solid harmonic of positive degree, it is more convenient to use the simplified form (16), and in the case of a tesseral solid harmonic, the results of differentiation give rise to a series of very interesting theorems, which, however, do not interest us at present.

It will be observed that the expressions given for F, G, H satisfy the equation of no convergence

$$\frac{d\mathrm{F}}{dx}+\frac{d\mathrm{G}}{dy}+\frac{d\mathrm{H}}{dz}=0.$$

(*b*.) When we use semipolar coordinates ρ, ϕ, z, given by

$$x=\rho\cos\phi,\ y=\rho\sin\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

the equations (13) transform into

$$\left.\begin{aligned}\frac{d\mathrm{H}}{\rho d\phi}-\frac{d.\rho\mathrm{G}}{\rho dz}&=-\frac{d\Omega}{d\rho}\\ \frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{d\rho}&=-\frac{d\Omega}{\rho d\phi}\\ \frac{d.\rho\mathrm{G}}{\rho d\rho}-\frac{d\mathrm{F}}{\rho d\phi}&=-\frac{d\Omega}{dz}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots \quad (18)$$

Here F, G, H are now the resolved components of F, G, H in the ρ—, ϕ—, z— directions, and the equations are obtained by resolving the two sides of equations (13) in these directions. The expressions on the left hand sides of the above equations are most readily recognised by observing that, when they vanish, we must have

$$[\mathrm{F}dx+\mathrm{G}dy+\mathrm{H}dz=]\,\mathrm{F}d\rho+\mathrm{G}\rho d\phi+\mathrm{H}dz$$

an exact differential. The equations themselves are satisfied by

$$\left.\begin{aligned}&\Omega=\frac{d\mathrm{P}}{dz},\quad \nabla^2\mathrm{P}=0,\\ &\mathrm{F}=-\frac{1}{\rho}\frac{d\mathrm{P}}{d\phi},\ \mathrm{G}=+\frac{d\mathrm{P}}{d\rho},\ \mathrm{H}=0\end{aligned}\right\}\quad\ldots\ldots\ldots \quad (19)$$

and the condition of no convergence

$$\frac{1}{\rho}\cdot\frac{d.\rho\mathrm{F}}{d\rho}+\frac{d\mathrm{G}}{\rho d\phi}+\frac{d\mathrm{H}}{dz}=0,\text{ is satisfied.}$$

(*c.*) If we employ polar coordinates r, θ, ϕ

$$[\mathrm{F}dx+\mathrm{G}dy+\mathrm{H}dz]=\mathrm{F}dr+\mathrm{G}rd\theta+\mathrm{H}.r\sin\theta d\phi$$

and the results of transforming the coordinates are obviously

$$\left.\begin{aligned}\frac{1}{r^2\sin\theta}\left(\frac{d.r\sin\theta\mathrm{H}}{d\theta}-\frac{d.r\mathrm{G}}{d\phi}\right)&=-\frac{d\Omega}{dr}\\ \frac{1}{r\sin\theta}\left(\frac{d\mathrm{F}}{d\phi}-\frac{d.r\sin\theta\mathrm{H}}{dr}\right)&=-\frac{d\Omega}{rd\theta}\\ \frac{1}{r}\left(\frac{d.r\mathrm{G}}{dr}-\frac{d\mathrm{F}}{d\theta}\right)&=-\frac{d\Omega}{r\sin\theta d\phi}\end{aligned}\right\}\quad\ldots\ldots\ldots \quad (20)$$

These equations are satisfied by

$$\left.\begin{array}{l}\Omega=\frac{d}{dr}(r\mathrm{P}),\ \nabla^2\mathrm{P}=0\\ \mathrm{F}=0,\ \mathrm{G}=-\frac{d\mathrm{P}}{\sin\theta d\phi},\ \mathrm{H}=+\frac{d\mathrm{P}}{d\theta}\end{array}\right\}\quad\ldots\ldots\ldots\quad(21)$$

and may be readily verified by substitution and actual differentiation, as in the former cases, observing that $\nabla^2\mathrm{P}=0$. They coincide with the expressions given by MAXWELL (Vol. II., Art. 671), and before him by JOCHMANN; they likewise satisfy the condition of no convergence, which in this case is

$$\frac{1}{r^2}\frac{d.\mathrm{F}r^2}{dr}+\frac{1}{r\sin\theta}\frac{d.\sin\theta\mathrm{G}}{d\theta}+\frac{d\mathrm{H}}{r\sin\theta d\phi}=0.$$

(B.) The vector components of the electro-magnetic momentum give rise to the vector potential of the magnetic force in the space in which the currents themselves exist: we shall therefore enquire what distribution of currents must be assigned that the vector potentials due to them may be respectively of the foregoing types.

(*a.*) *Rectangular coordinates.*

The equations to be satisfied are

$$4\pi u=\frac{d\gamma}{dy}-\frac{d\beta}{dz},\ \ldots$$

$$\alpha=\frac{d\mathrm{H}}{dy}-\frac{d\mathrm{G}}{dz},\ \ldots$$

If we take

$$\mathrm{F}=-\frac{d\mathrm{P}}{dy},\ \mathrm{G}=\frac{d\mathrm{P}}{dx},\ \mathrm{H}=0\quad\ldots\ldots\ldots\quad(22_1)$$

we find that the above equations are satisfied by

$$\left.\begin{array}{l}\alpha=-\frac{d^2\mathrm{P}}{dxdz},\ \beta=-\frac{d^2\mathrm{P}}{dydz},\ \gamma=\frac{d^2\mathrm{P}}{dx^2}+\frac{d^2\mathrm{P}}{dy^2}\\ u=-\frac{d\Phi}{dy},\ v=\frac{d\Phi}{dx},\ w=0\end{array}\right\}\quad\ldots\ldots\quad(22_2)$$

where

$$4\pi\Phi=-\nabla^2\mathrm{P}.$$

Φ is here the current function, and $\nabla^2\mathrm{P}$ is not now supposed to be equal to zero; in fact, nothing is at present supposed to be known about it.

(*b.*) *Semipolar coordinates.*

The x— and y— directions are variable and are supposed to be in the directions of

$d\rho$ and $d\phi$ respectively, the directed quantities, as above, preserving their former designations.

Our equations are

$$4\pi u=\frac{1}{\rho}\left(\frac{d\gamma}{d\phi}-\frac{d\rho\beta}{dz}\right),\quad 4\pi v=\frac{d\alpha}{dz}-\frac{d\gamma}{d\rho},\quad 4\pi w=\frac{1}{\rho}\left(\frac{d\rho\beta}{d\rho}-\frac{d\alpha}{d\phi}\right)$$

$$\alpha=\frac{1}{\rho}\left(\frac{d\mathrm{H}}{d\phi}-\frac{d\rho\mathrm{G}}{dz}\right),\quad \beta=\frac{d\mathrm{F}}{dz}-\frac{d\mathrm{H}}{d\rho},\quad \gamma=\frac{1}{\rho}\left(\frac{d\rho\mathrm{G}}{d\rho}-\frac{d\mathrm{F}}{d\phi}\right),$$

and we may satisfy these by

$$\mathrm{F}=-\frac{1}{\rho}\frac{d\mathrm{P}}{d\phi},\quad \mathrm{G}=\frac{d\mathrm{P}}{d\rho},\quad \mathrm{H}=0 \quad \ldots\ldots\ldots\ldots \quad (23_1)$$

and

$$\left.\begin{aligned}&\alpha=-\frac{d^2\mathrm{P}}{d\rho dz},\quad \beta=-\frac{1}{\rho}\frac{d^2\mathrm{P}}{d\phi dz},\quad \gamma=\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d\mathrm{P}}{d\rho}\right)+\frac{1}{\rho^2}\frac{d^2\mathrm{P}}{d\phi^2}\\ &u=-\frac{1}{\rho}\frac{d\Phi}{d\phi},\quad v=\frac{d\Phi}{d\rho},\quad w=0\end{aligned}\right\} \quad \ldots\ldots \quad (23_2)$$

$$4\pi\Phi=-\nabla^2\mathrm{P}\,;$$

here

$$\nabla^2=\frac{1}{\rho}\frac{d}{d\rho}\cdot\rho\frac{d}{d\rho}\cdot+\frac{1}{\rho^2}\frac{d^2}{d\phi^2}\cdot+\frac{d^2}{dz^2}\cdot$$

(*c.*) *Polar coordinates.*

The equations in this case are, resolving along dr, $rd\theta$, $r\sin\theta d\phi$,

$$4\pi u=\frac{1}{r\sin\theta}\left(\frac{d\gamma\sin\theta}{d\theta}-\frac{d\beta}{r\phi}\right),\quad 4\pi v=\frac{1}{r\sin\theta}\left(\frac{d\alpha}{d\phi}-\frac{dr\gamma\sin\theta}{dr}\right),\quad 4\pi w=\frac{1}{r}\left(\frac{d.r\beta}{dr}-\frac{d\alpha}{d\theta}\right)$$

$$\alpha=\frac{1}{r\sin\theta}\left(\frac{d\mathrm{H}\sin\theta}{d\theta}-\frac{d\mathrm{G}}{d\phi}\right),\ldots=\ldots \qquad \ldots=\ldots \qquad ;$$

and we may put in these

$$\mathrm{F}=0,\quad \mathrm{G}=-\frac{1}{\sin\theta}\frac{d\mathrm{P}}{d\phi},\quad \mathrm{H}=\frac{d\mathrm{P}}{d\theta}\cdot \quad \ldots\ldots\ldots\ldots \quad (24_1)$$

$$\left.\begin{aligned}&\alpha=+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\cdot\sin\theta\frac{d\mathrm{P}}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2\mathrm{P}}{d\phi^2}\right],\quad \beta=-\frac{1}{r}\frac{d^2(\mathrm{P}r)}{drd\theta},\quad \gamma=-\frac{1}{r\sin\theta}\frac{d^2(\mathrm{P}r)}{drd\phi}\cdot\\ &u=0,\quad v=-\frac{d\Phi}{\sin\theta d\phi},\quad w=+\frac{d\Phi}{d\theta}\end{aligned}\right\} \cdot \quad (24_2)$$

where

$$4\pi\Phi=-\nabla^2\mathrm{P}.$$

General problem of induction.

§ 4. The problems to which we shall chiefly devote our attention in what follows are the induction of currents in a plate (i.) of finite and (ii.) of infinitesimal thickness, either at rest or revolving uniformly about an axis normal to its faces, and in a sphere or spherical shell either of finite or infinitesimal thickness under the same conditions.

We may observe however, preliminarily, that *any* solution of any problem of induction which satisfies all the conditions of the question is the only solution that can exist. For all the equations being linear the difference between the functions which express two solutions would also satisfy all the differential equations, and would correspond to the case in which the conductor is under no external inducing forces, and therefore no currents can be set up in it.

This observation has been made by MAXWELL and has been established more in detail by HELMHOLTZ in his memoir. From the linearity of the equations we may also conclude that the effect of different systems acting either simultaneously or in succession may be found by adding the effects due to each separately.

In particular, when the conductor is at rest, its state at any time t may be considered as the aggregate effect produced by a continuous series of impulses during its previous history. The effect of each of these impulses, after being received, decays according to a certain law, and each contributes a certain amount to the total result at any time. The characteristic equation in these investigations is

$$\frac{\sigma}{4\pi}\nabla^2 P=\frac{d}{dt}(P+P_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where P_0 is due to the direct action of the external electro-magnets.

Let an impulsive change take place in this external system, by which P_0 suddenly rises from 0 to P'_0, and let P increase at the same time from 0 to P'; then, integrating the above equation during the impulse, we obtain

$$P'+P'_0=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The impulse having been administered, P decays according to a law peculiar to each system, and satisfies during its decay the equation

$$\frac{\sigma}{4\pi}\nabla^2 P=\frac{dP}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Let $P_0(t)$ be the value of P_0 at any time t: this value may be supposed to have been accumulated by impulses during the previous history of the system, and, if τ be the time measured backwards from t, we may express this fact by the equation

$$P_0=-\int_0^\infty \frac{dP_0(t-\tau)}{d\tau}d\tau.$$

Let $\chi_t(\mathrm{P}_0)$ represent the law of decay of a system which was initially represented by P_0; then, combining equations 26 and 27, we see that the complete value of P at any time for the system is given by

$$\mathrm{P}=+\int_0^\infty \chi_\tau\left(\frac{d\mathrm{P}_0(t-\tau)}{d\tau}\right).\,d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

To complete the solution, therefore, we have only to determine the nature of χ_τ.

Case of an infinite plate of finite thickness.

§ 5. Let us now take up the case of an infinitely extended plane plate of thickness $2b$, and suppose the origin somewhere midway between the two faces of the plate so that its faces are determined by $z=\pm b$.

The scalar and vector potentials of external magnets or currents may be denoted by

$$\Omega_0=\frac{d\mathrm{P}_0}{dz},\ \mathrm{F}_0=-\frac{d\mathrm{P}_0}{dy},\ \mathrm{G}_0=\frac{d\mathrm{P}_0}{dx},\ \mathrm{H}_0=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The equations of the currents in the plate will have for their type

$$\sigma u=-\frac{d(\mathrm{F}+\mathrm{F}_0)}{dt}-\frac{d\Psi}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and we shall prove that all the conditions of the problem may be satisfied by taking

$$\left.\begin{array}{l} \Psi=0,\ \mathrm{F}=-\dfrac{d\mathrm{P}}{dy},\ \mathrm{G}=\dfrac{d\mathrm{P}}{dx},\ \mathrm{H}=0 \\ u=-\dfrac{d\Phi}{dy},\ v=\dfrac{d\Phi}{dx},\ w=0 \\ \dfrac{\sigma}{4\pi}\Phi=-\nabla^2\mathrm{P} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The equations of electromotive force are now reduced to the single characteristic equation

$$\frac{\sigma}{4\pi}\nabla^2\mathrm{P}=\frac{d}{dt}(\mathrm{P}+\mathrm{P}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Since there is no free electricity anywhere present, the currents in the plate are closed currents, and therefore F, G, H and their differential coefficients are all continuous in passing across the boundary of the conductor. All these conditions will be satisfied by having

$$\left.\begin{aligned} \mathrm{P}_{\text{outside}} &= \mathrm{P}_{\text{inside}} \\ \left(\frac{d\mathrm{P}}{dz}\right)_{\text{outside}} &= \left(\frac{d\mathrm{P}}{dz}\right)_{\text{inside}} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

when $z=\pm b$; and it must be remembered that, outside the plate, the vector potential is determined by

$$\mathrm{F}=-\frac{d\mathrm{P}}{dy},\ \mathrm{G}=\frac{d\mathrm{P}}{dx},\ \mathrm{H}=0,\ \nabla^2\mathrm{P}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

If we employ semipolar coordinates the equation in P in the substance of the plate may be written

$$\frac{\sigma}{4\pi}\left(\frac{d^2\mathrm{P}}{d\rho^2}+\frac{1}{\rho}\frac{d\mathrm{P}}{d\rho}+\frac{1}{\rho^2}\frac{d^2\mathrm{P}}{d\phi^2}+\frac{d^2\mathrm{P}}{dz^2}\right)=\frac{d\mathrm{P}}{dt}.$$

To satisfy it, put

$$\mathrm{P}=\cos m\phi \mathrm{J}_m(\kappa\rho)(\mathrm{A}\cos nz+\mathrm{B}\sin nz)e^{-\lambda t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where $\mathrm{J}_m(\kappa\rho)$ is BESSEL's function of the m^{th} degree satisfying the equation

$$\frac{d^2\mathrm{J}}{d\rho^2}+\frac{1}{\rho}\frac{d\mathrm{J}}{d\rho}+\left(\kappa^2-\frac{m^2}{\rho^2}\right)\mathrm{J}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

and

$$\lambda=\frac{\sigma}{4\pi}(\kappa^2+n^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

We observe also that κ is a constant, which the problem does not enable us to determine: it must therefore be supposed to have all values from 0 to ∞; m is necessarily a positive integer.

Outside the plate P is given by $\nabla^2\mathrm{P}=0$, and is satisfied by

$$\left.\begin{aligned} \mathrm{P}&=e^{-\lambda t}\cos m\phi \mathrm{J}_m(\kappa\rho).\mathrm{C}e^{-\kappa z},\ z \text{ positive} \\ \mathrm{P}&=e^{-\lambda t}\cos m\phi \mathrm{J}_m(\kappa\rho).\mathrm{D}e^{+\kappa z},\ z \text{ negative} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

it being observed that P must vanish when $z=\infty$.

To determine n, we have, by equations (33),

(1) when $z=+b$.

$$\begin{aligned} \mathrm{A}\cos nb+\mathrm{B}\sin nb &= \mathrm{C}e^{-\kappa b} \\ -n(\mathrm{A}\sin nb-\mathrm{B}\cos nb) &= -\kappa\mathrm{C}e^{-\kappa b}, \end{aligned}$$

(2) when $z=-b$

$$\begin{aligned} \mathrm{A}\cos nb-\mathrm{B}\sin nb &= \mathrm{D}e^{-\kappa b} \\ +n(\mathrm{A}\sin nb+\mathrm{B}\cos nb) &= \kappa\mathrm{D}e^{-\kappa b}. \end{aligned}$$

Eliminating C and D,

$$\mathrm{A}(\kappa\cos nb-n\sin nb)+\mathrm{B}(\kappa\sin nb+n\cos nb)=0$$
$$\mathrm{A}(\kappa\cos nb-n\sin nb)-\mathrm{B}(\kappa\sin nb+n\cos nb)=0.$$

From these we find

$$\left.\begin{array}{l}(1)\ \mathrm{B}=0,\ n\sin nb-\kappa\cos nb=0,\ \mathrm{C}=\mathrm{A}e^{\kappa b}\cos nb=\mathrm{D}\\ (2)\ \mathrm{A}=0,\ \kappa\sin nb+n\cos nb=0,\ \mathrm{C}=\mathrm{B}e^{\kappa b}\cos nb=-\mathrm{D}\end{array}\right\} \quad . \quad . \quad . \quad (39)$$

Putting all these together we obtain

$$\left.\begin{array}{ll}\text{inside the plate} & \mathrm{P}=\Sigma^{(3)}\cos m\phi \mathrm{J}_m(\kappa\rho)(\mathrm{A}\cos nze^{-\lambda t}+\mathrm{B}\sin n'ze^{-\lambda' t})\\ & \qquad\qquad\qquad\qquad +\text{similar terms in }\sin m\phi\\ \text{outside, } z+^{\text{ve}}, & \mathrm{P}=\Sigma^{(3)}\cos m\phi \mathrm{J}_m(\kappa\rho)(\mathrm{A}\cos nbe^{-\lambda t}+\mathrm{B}\sin n'be^{-\lambda' t})e^{-\kappa(z-b)}+\,.\,.\\ \text{outside, } z-^{\text{ve}}, & \mathrm{P}=\Sigma^{(3)}\cos m\phi \mathrm{J}_m(\kappa\rho)(\mathrm{A}\cos nbe^{-\lambda t}-\mathrm{B}\sin n'be^{-\lambda' t})e^{\kappa(z+b)}+\,.\,.\end{array}\right\} \quad (40)$$

where

$$n\sin nb-\kappa\cos nb=0,\ \lambda=\frac{\sigma}{4\pi}(\kappa^2+n^2)$$
$$n'\cos n'b+\kappa\sin n'b=0,\ \lambda'=\frac{\sigma}{4\pi}(\kappa^2+n'^2)$$

The summations are to be extended, first over all the values of n and n' corresponding to the roots of the above equations, then over all values of κ from 0 to ∞, and finally over all integral values of m from 0 to ∞; the summation with respect to κ will be of the nature of an integral, as will presently appear.

§ 6. The investigation of the values of the coefficients is attended with some difficulty owing to the difficulty of interpreting the values of J_m for infinite values of the argument. I have therefore sought to evade these difficulties by conceiving the plate as the limit of a spherical shell of finite thickness but of infinite radius, and keeping in view the general course which the solution for a spherical shell takes. It is possible to obtain the solution for any spherical shell, and it might seem therefore easy at once to adapt that solution to the present case: but unfortunately the adaptation is also beset with difficulties of a peculiar kind, and therefore we can do no more than take the steps of that investigation as guides in the present problem. The main light which the case of the spherical shell gives us is that we have to regard $\cos m\phi \mathrm{J}_m(\kappa\rho)$ as a degraded form of the spherical surface-harmonic $\cos m\phi \mathrm{P}_m{}^n(\cos\theta)$, obtained by putting

$$\sin\theta=\frac{\rho}{\mathrm{a}},\ n=\kappa\mathrm{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}_1)$$

a being the mean radius of the shell (sensibly constant), κ a finite quantity: to make the passage clear we know that, if $s=\sin\theta$, $\mathrm{P}_m{}^n$ satisfies the equation

$$(1-s^2)\frac{d^2y}{ds^2}+\frac{1-2s^2}{s}\frac{dy}{ds}+\left\{\left(n(n+1)-\frac{m^2}{s^2}\right)\right\}y=0 \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}_2)$$

(see HEINE, 'Kugelfunctionen,' p. 216, 2nd ed.); which we may satisfy by

$$y=\mathrm{C}_m{}^n(s^m+\mathrm{A}_1s^{m+2}+\mathrm{A}_2s^{m+4}+\ldots)$$

where

$$\mathrm{A}_1\{(m+2)^2-m^2\}+n(n+1)-m.(m+1)=0$$
$$\mathrm{A}_2\{(m+4)^2-m^2\}+[n(n+1)-(m+2)(m+3)]\mathrm{A}_1=0$$
$$\mathrm{A}_3\{(m+6)^2-m^2\}+[n.(n+1)-(m+4)(m+5)]\mathrm{A}_2=0, \text{ \&c.}$$

If we now put n (infinitely great) $=\kappa\mathrm{a}$, $s=\frac{\rho}{\mathrm{a}}$, we find

$$\mathrm{P}_m{}^n=\frac{\mathrm{C}_m{}^n}{\mathrm{a}^m}\rho^m\left(1-\frac{\kappa^2\rho^2}{4.\overline{m+1}}+\frac{\kappa^4\rho^4}{4.8.\overline{m+1}.\overline{m+2}}-\frac{\kappa^6\rho^6}{4.8.12.\overline{m+1}.\overline{m+2}.\overline{m+3}}+\ldots\right)$$
$$=\frac{2^m m!}{\kappa^m\mathrm{a}^m}\mathrm{C}_m{}^n.\mathrm{J}_m(\kappa\rho) \quad . \quad (\mathrm{A}_3)$$

the value of the constant $\mathrm{C}_m{}^n$ being obtained from considering the value of $\mathrm{P}_m{}^n\div s^m$ when $s=0$; in fact we have (HEINE, 2nd ed., p. 207),

$$\mathrm{C}_m{}^n=2^{n-m}\frac{n!(n+m)!}{m!(2n)!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}_4)$$

This constant we shall keep, for the present, unreduced.

From the theorem $\int_0^\pi \mathrm{P}_m{}^n.\mathrm{P}_m{}^{n'}\sin\theta d\theta=0$, we derive at once

$$\int_0^\infty \mathrm{J}_m(\kappa\rho)\mathrm{J}_m(\kappa'\rho)\rho d\rho=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}_5)$$

We have moreover (see HEINE, pp. 327 and 253),

$$\int_0^\pi(\mathrm{P}_m{}^n)^2\sin\theta d\theta=\frac{2}{2n+1}.\frac{(n+m)!(n-m)!}{(1.3.5\ldots\overline{2n-1})^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}_6)$$

and, correspondingly,

$$\int_0^\infty\overline{\mathrm{J}_m(\kappa\rho)}|^2\rho d\rho=\frac{\kappa^{2m}\mathrm{a}^{2m+2}}{(\mathrm{C}_m{}^n)^2}.\frac{2}{2n+1}.\frac{(n+m)!(n-m)}{(1.3.5\ldots 2n-1)^2}.\frac{1}{2^{2m}(m!)^2}$$

If we take into account that n is infinitely great compared to m, we see that $(n-m)!=n!\div n^m$ and $(n+m)!=n!\times n^m$; also

$$\frac{1}{\mathrm{C}_m{}^n}=2^m m!\frac{1.3.5\ldots(2n-1)}{n!\,n^m},$$

whence

$$\int_0^\infty \overline{\mathrm{J}_m(\kappa\rho)}|^2\rho d\rho=\frac{\kappa^{2m}\mathrm{a}^{2m+2}}{n^{2m}}\cdot\frac{1}{n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}_7)$$

But $n=\kappa\mathrm{a}$ and the successive values of n correspond to the successive integers $\therefore\ \delta\kappa=\frac{1}{\mathrm{a}}$; hence the above integral becomes

$$\int_0^\infty \overline{\mathrm{J}_m(\kappa\rho)}|^2\rho d\rho=\frac{1}{\kappa\delta\kappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A}_8)$$

. We are now in a position to find the values of the coefficients (A), (B) in the expressions (40): for it may be easily shown that, if n_1, n_2 be two different values of n, n'_1, n'_2 two different values of n' from equations (39), then

$$\left.\begin{array}{c}\displaystyle\int_{-b}^{+b}\cos n_1z\cos n_2zdz=0,\quad \int_{-b}^{+b}\cos nz\sin n'zdz=0,\quad \int_{-b}^{+b}\sin n'_1z\sin n'_2zdz=0 \\ \displaystyle\int_{-b}^{+b}\cos^2 nzdz=\frac{2nb+\sin 2nb}{2n},\quad \int_{-b}^{+b}\sin^2 n'zdz=\frac{2n'b-\sin 2n'b}{2n'}\end{array}\right\} \quad . \quad (41)$$

From these results we may separate the different values of A, B: for since when $t=0$, $\mathrm{P}=-\mathrm{P}_0$.

We have in fact

$$\mathrm{A}=-\frac{1}{\pi}\cdot\frac{2n}{2nb+\sin 2nb}\cdot\kappa\delta\kappa\,.\int_0^{2\pi}\cos m\phi d\phi\int_0^\infty\rho\mathrm{J}_m(\kappa\rho)d\rho\int_{-b}^{+b}\mathrm{P}_0\cos nzdz \quad . \quad . \quad (42)$$

B may be found in a similar manner, and likewise also the terms depending on sin $m\phi$. Collecting all these results for the value of P (ϕ, ρ, z) at an external point on the positive side of the plate, we have (z positive)

$$\mathrm{P}=-\frac{1}{\pi}\Sigma_m\int_0^{2\pi}\cos m(\phi-\phi')d\phi'\int_0^\infty\kappa e^{-\kappa(z-b)}\mathrm{J}_m(\kappa\rho)d\kappa\int_0^\infty\rho'\mathrm{J}_m(\kappa\rho')d\rho'\Sigma$$

$$\left\{e^{-\lambda t}\frac{2n\cos nb}{2nb+\sin 2nb}\int_{-b}^{+b}\mathrm{P}'_0\cos nz'dz'+e^{-\lambda' t}\cdot\frac{2n'\sin n'b}{2n'b-\sin 2n'b}\int_{-b}^{+b}\mathrm{P}'_0\sin n'z'dz'\right\} \quad . \quad . \quad (43)$$

The last Σ is to be extended over all the values of n, n' which can be derived as roots of the equations in these quantities. The first Σ denotes that the summation is to be extended to all values of m. It is to be feared there is no simpler mode of writing the above result; but the approximate solution of any given case could be obtained from it by carefully conducting the approximations.

Case of an indefinitely thin plane at rest.

§ 7. We may pass from the above problem to the present by making b indefinitely small. If R be the resistance to the current per unit of area

$$2\mathrm{R}b=\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and we have also to enquire what the values of n and n' derived—

(*a.*) From the former solution will now become. To do so I shall write the equation for n in the form

$$\theta \sin \theta-\kappa b \cos \theta=0, \text{ where } \theta=nb.$$

(1.) When κb is small the roots of the equation are approximated to, *in general,* by $\sin \theta=0$, or $\theta=j\pi=\beta$, say.

Let $\theta=\beta+x$, then to find x we have

$$(\beta+x) \sin x=\kappa b \cos x.$$

Expanding $\sin x$ and $\cos x$ in ascending powers of x, we obtain, after some reduction, x in the following series of powers of κb

$$x=\frac{1}{\beta}.\kappa b-\frac{1}{\beta^3}\kappa^2b^2-\frac{6+\beta^2}{5\beta^5}\kappa^3b^3+ \ldots$$

moreover

$$\lambda=\frac{\mathrm{R}b}{2\pi}(\kappa^2+n^2).$$

Hence, *in general,*

$$\left.\begin{aligned} nb&=j\pi+\frac{1}{j\pi}\kappa b-\frac{1}{j^3\pi^3}\kappa^2b^2-\frac{6+j^2\pi^2}{3j^5\pi^5}\kappa^3b^3+ \ldots \\ \lambda&=\frac{\mathrm{R}}{2\pi b}\left(j^2\pi^2+2\kappa b+\frac{j^2\pi^2-1}{j^2\pi^2}\kappa^2b^2-\frac{18+2j^2\pi^2}{3j^4\pi^4}.\kappa^3b^3+ \ldots\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (45)$$

(2.) But there is a root of the equation for which $n\,b$ is very small given approximately by $n^2b^2=\kappa b$; expanding fully the equation $\theta \sin \theta-\kappa b \cos \theta=0$ in ascending powers of θ, we find

$$\left.\begin{aligned} n^2b^2&=\kappa b-\tfrac{1}{3}\kappa^2b^2+\tfrac{4}{45}\kappa^3b^3+ \ldots \\ \lambda&=\frac{\mathrm{R}}{2\pi}(\kappa+\tfrac{2}{3}\kappa^2b+\tfrac{4}{45}\kappa^3b^2+ \ldots) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

(*b.*) In like manner, the equation in n', which is

$$n'b\cos n'b+\kappa b\sin n'b=0,$$

is satisfied, approximately, by $n'b=\frac{2i+1}{2}.\pi=\alpha$, say: and more accurately by $n'b=\alpha+x$, where

$$(\alpha+x)\sin x=\kappa b\cos x.$$

This is precisely the same equation as before, and gives

$$\left.\begin{aligned} n'b&=\alpha+\frac{1}{\alpha}.\kappa b-\frac{1}{\alpha^3}\kappa^2b^2-\frac{6+\alpha^2}{3\alpha^5}\kappa^3b^3 \\ \lambda'&=\frac{\mathrm{R}}{2\pi b}\left(\alpha^2+2\kappa b+\frac{\alpha^2-1}{\alpha^2}\kappa^2b^2-\frac{18+2\alpha^2}{3\alpha^4}\kappa^3b^3+\ \ldots\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (47)$$

When we make b indefinitely small, retaining R as a finite quantity, all the values of λ and λ' become indefinitely great except those which arise from the approximation in n, $n^2b^2=\kappa b+\ldots$; λ is then $=\frac{\mathrm{R}\kappa}{2\pi}$. Rejecting therefore all other values of n, n', λ and λ', and confining our attention to the parts of P outside the plate, we find [writing A cos nb under the single term $\mathfrak{A}$],

$$\left.\begin{aligned} z\text{ positive, }\ \mathrm{P}&=\Sigma\cos m\phi.\mathrm{J}^m(\kappa\rho).\mathfrak{A}e^{-\kappa\left(z+\frac{\mathrm{R}t}{2\pi}\right)} \\ z\text{ negative, }\ \mathrm{P}&=\Sigma\cos m\phi.\mathrm{J}^m(\kappa\rho).\mathfrak{A}e^{+\kappa\left(z-\frac{\mathrm{R}t}{2\pi}\right)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

the term in b in the exponential e being neglected.

If, at the surface of the plate, $\mathrm{P}=f(x,y,t)$, then at any point for which z is positive,

$$\left.\begin{aligned} &\mathrm{P}=f\left(x,y,t+\frac{2\pi z}{\mathrm{R}}\right) \\ \text{and at any point, } z \text{ negative,}\quad &\mathrm{P}=f\left(x,y,t-\frac{2\pi z}{\mathrm{R}}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

This is MAXWELL's result.

I have dwelt somewhat fully upon this case because it brings out very clearly the mode in which the terms depending upon all but a certain number of the values of λ and λ' disappear when b is made indefinitely small. When we consider the case of a spherical shell, we shall find that a similar reduction takes place.

Induction of currents in a solid sphere at rest.

§ 8. When a solid sphere of radius a is influenced by an external magnetic system whose potential is Ω_0, we may express the vector potential of the latter by taking

$$\Omega_0=\frac{d}{dr}(P_0 r),\ F_0=0,\ G_0=-\frac{dP_0}{\sin\theta d\phi},\ H_0=\frac{dP_0}{d\theta}\ \ldots\ldots\ \tag{50}$$

dx, dy, dz being now coincident with dr, $rd\theta$, $r\sin\theta d\phi$.

The equations of electromotive force, viz.:

$$\left.\begin{aligned}\sigma u&=-\frac{d(F+F_0)}{dt}-\frac{d\psi}{dr}\\ \sigma v&=-\frac{d(G+G_0)}{dt}-\frac{d\psi}{rd\theta}\\ \sigma w&=-\frac{d(H+H_0)}{dt}-\frac{d\psi}{r\sin\theta d\phi}\end{aligned}\right\}\ \ldots\ldots\ \tag{51}$$

and all the conditions of the problem may be satisfied by taking

$$\left.\begin{aligned}&\psi=0,\ F=0,\ G=-\frac{dP}{\sin\theta d\phi},\ H=\frac{dP}{d\theta}\\ &u=0,\ v=-\frac{d\Phi}{\sin\theta d\phi},\ w=\frac{d\Phi}{d\theta}\\ &4\pi\Phi=-\nabla^2 P.\end{aligned}\right\}\ \ldots\ldots\ \tag{52}$$

P satisfies the characteristic equation

$$\frac{\sigma}{4\pi}\nabla^2 P=\frac{dP}{dt},$$

or, more fully,

$$\frac{d^2P}{dr^2}+\frac{2}{r}\frac{dP}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{d^2P}{d\phi^2}=\frac{4\pi}{\sigma}\frac{dP}{dt};\ \ldots\ \tag{53}$$

outside the sphere P satisfies the equation

$$\nabla^2 P=0;$$

and we must have the further condition that F, G, H and their differential coefficients, or (which is the same thing)

$$P \text{ and } \frac{dP}{dr}\ \ldots\ldots\ldots\ \tag{54}$$

are to undergo no sudden change in passing across the surface of the sphere, and that P must be finite at the centre.

The equation in P (inside the sphere), is satisfied by

$$P=AU_nS_n(\lambda r)e^{-\lambda^2\cdot\frac{\sigma t}{4\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

where U_n is any surface harmonic of degree n, and $S_n(\lambda r)$ is the solution of the equation

$$\frac{d^2Q}{dr^2}+\frac{2}{r}\frac{dQ}{dr}+\left(\lambda^2-\frac{n(n+1)}{r^2}\right)Q=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

which does not become infinite at the centre. In fact the two solutions of this equation are, as I have elsewhere shown

$$\left.\begin{aligned} S_n(\lambda r)&=(\lambda r)^n\cdot\left(\frac{1}{\lambda r}\frac{d}{d\lambda r}\right)^n\cdot\frac{\sin\lambda r}{\lambda r} \\ T_n(\lambda r)&=(\lambda r)^n\left(\frac{1}{\lambda r}\frac{d}{d\lambda r}\right)^n\frac{\cos\lambda r}{\lambda r} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_1)$$

of which the former vanishes at the centre, except when $n=0$, and the latter vanishes at ∞; it is the former of these expressions which is chosen.

Outside the sphere the value of P is

$$P=BU_n.r^{-n-1}e^{-\lambda^2\cdot\frac{\sigma t}{4\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

and the above boundary conditions give

$$A_nS_n(\lambda a)=B_na^{-n-1},\ A_n\frac{dS_n}{da}=-B_n(n+1)a^{-n-2},$$

or

$$B=a^{n+1}.A_nS(\lambda a)$$

$$a\frac{dS_n}{da}+(n+1)S_n=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

This equation is also equivalent to the following somewhat simpler one (see equation B_5, § 10),

$$S_{n-1}(\lambda a)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57_1)$$

The values of λ being thus determined, we may proceed as in the former problem to find the coefficients due to any initial circumstances, and thence the state of the sphere may be readily deduced by the principles which have been already explained. I may observe that U_n is to be expressed in the system of tesseral harmonics

$$\sum_0^n (A \cos m\phi + B_m \sin m\phi) P_m{}^n$$

The rest of the process needs no remark.

Spherical shell of finite thickness.

§ 9. Let the external and internal radii be b, a; all the conditions of the problem will be satisfied by the same general assumptions as to the forms of the electromagnetic momentum of the currents, the current function and of the potentials of the external magnetism. The characteristic equation remains

$$\frac{\sigma}{4\pi}\nabla^2 P = \frac{dP}{dt};$$

and, as in the former cases, since there is nowhere flow along the radii to the common centre of the surfaces and no free electricity, the currents are entirely confined to the interior of the shell, and we shall have as before for boundary conditions, that the vector potential of the currents in the shell and its differential coefficients sustain no sudden change at crossing the surfaces. Outside the shell and within its inner surface, the vector potential is given by expressions of the same form as in the substance, viz.:

$$F=0, \quad G=-\frac{dP}{\sin\theta d\phi}, \quad H=\frac{dP}{d\theta},$$

but in the space free from currents

$$\nabla^2 P = 0.$$

At the boundaries all the conditions are satisfied by having

$$P \text{ and } \frac{dP}{dr}$$

continuous when $r=a$ and when $r=b$; we must therefore take

$$\left.\begin{array}{ll} \text{within sphere of radius a,} & P=\Sigma C r^n U_n e^{-\lambda^2 \frac{\sigma}{4\pi}.t} \\ \text{in substance of shell,} & P=\Sigma.(AS_n+BT_n)U_n e^{-\lambda^2.\frac{\sigma t}{4\pi}} \\ \text{without sphere of radius b,} & P=\Sigma D r^{-n-1}.U_n e^{-\lambda^2.\frac{\sigma t}{4\pi}} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (58)$$

S_n and T_n are the expressions referred to in last article: they are the particular solutions of the equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\lambda^2-\frac{n(n+1)}{r^2}\right)R=0.$$

If we write $x=\lambda r$, S_n and T_n are respectively equal to

$$x^n.\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x} \text{ and } x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n.\frac{\cos x}{x}.$$

To express the boundary conditions we shall write

$$\alpha=\lambda a,\ \beta=\lambda b,\ S=S(\lambda a),\ S'=S(\lambda b),\ \&c\,; \quad \ldots\ldots \quad (59)$$

we have, accordingly,

$$\left.\begin{aligned} &Ca^n=AS+BT,\ nCa^{n-1}=\lambda\left(A\frac{dS}{d\alpha}+B\frac{dT}{d\alpha}\right)\\ &Db^{-n-1}=AS'+BT', -(n+1)Db^{-n+2}=\lambda\left(A\frac{dS'}{d\beta}+B\frac{dT'}{d\beta}\right)\end{aligned}\right\}\quad \ldots \quad (60)$$

These equations may be put into much more elegant forms; but to do so we require various properties of the S— and T— functions; and as these are not very generally understood, I shall here digress into a brief sketch of the whole subject, confining myself as strictly as possible to those properties which have a direct bearing on the present question.

[*Properties of the "Spherical Functions."*

§ 10. If we write $x^2=t$, and work out the differentiations and integrations, it can be readily shown that

$$\left.\begin{aligned} x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\sin x}{x}&=(-1)^n\ \ x^{-n-1}.\left(\frac{1}{x}\frac{d}{dx}\right)^{-n-1}\frac{\cos x}{x}\\ x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\frac{\cos x}{x}&=(-1)^{n+1}x^{-n-1}\left(\frac{1}{x}\frac{d}{dx}\right)^{-n-1}\frac{\sin x}{x}\end{aligned}\right\}\quad \ldots\ldots \quad (B_3)$$

The constants introduced by integration must be so adjusted as to make a certain number of terms in the second equality coincide: the others will be exactly equal.

If we write each member of these two equations, as before, respectively S_n and T_n, and differentiate the quantities on the left-hand sides of the equations, we find

$$xS_{n+1}=x\frac{dS_n}{dx}-nS_n,\ xT_{n+1}=x\frac{dT_n}{dx}-nT_n \quad \ldots\ldots \quad (B\,)$$

Differentiating the right-hand members we obtain

$$-xS_{n-1}=x\frac{dS_n}{dx}+(n+1)S_n,\quad -xT_{n-1}=x\frac{dT_n}{dx}+(n+1)T_n \quad . \quad . \quad . \quad . \quad (B_5)$$

These are the fundamental formulæ of reduction in these functions: if in the latter we write $n+1$ for n, and substitute the values of S_{n+1} and T_{n+1} from the former, we can show easily that S_n and T_n satisfy the differential equation

$$\frac{d^2R}{dx^2}+\frac{2}{x}\frac{dR}{dx}+\left(1-\frac{n.n+1}{x^2}\right)R=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

and are therefore the two solutions of it. If we eliminate from the above $x\frac{dS_n}{dx}$, and $x\frac{dT_n}{dx}$, we find

$$\left.\begin{array}{l}(2n+1)\dfrac{S_n}{x}+S_{n+1}+S_{n-1}=0\\[2ex](2n+1)\dfrac{T_n}{x}+T_{n+1}+T_{n-1}=0\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_6)$$

whence also

$$S_{n+1}T_n-T_{n+1}S_n=S_nT_{n-1}-S_{n-1}T_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_7)$$

and, by successive reductions,

$$S_{n+1}T_n-T_{n+1}S_n=S_1T_0-T_1S_0=\frac{1}{x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_8)$$

and thus also, by (B_6),

$$S_{n+1}T_{n-1}-T_{n+1}S_{n-1}=-\frac{2n+1}{x^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_9)$$

We obtain also directly from equations (B_4),

$$T_n\frac{dS_n}{dx}-S_n\frac{dT_n}{dx}=S_{n+1}T_n-T_{n+1}S_n=\frac{1}{x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B_{10})$$

a well-known result.]

11. If we now turn to equations (60), we find by elimination

$$\lambda A\left(T\frac{dS}{d\alpha}-S\frac{dT}{d\alpha}\right)=-a^{n-1}C\left(\alpha\frac{dT}{d\alpha}-nT\right)$$

$$\lambda B\left(T\frac{dS}{d\alpha}-S\frac{dT}{d\alpha}\right)=+a^{n-1}C\left(\alpha\frac{dS}{d\alpha}-nS\right)$$

$$\lambda A\left(T'\frac{dS'}{d\beta}-S'\frac{dT'}{d\beta}\right)=-b^{-n-2}D\left(\beta\frac{dT'}{d\beta}+(n+1)T'\right)$$

$$\lambda B\left(T'\frac{dS'}{d\beta}-S'\frac{dT'}{d\beta}\right)=+b^{-n-2}D\left(\beta\frac{dS'}{d\beta}+(n+1)S'\right)$$

These may be written

$$\mathrm{A}=-\lambda^2 \mathrm{a}^{n+2}\mathrm{T}_{n+1}(\lambda \mathrm{a})\mathrm{C}, \qquad \mathrm{B}=+\lambda^2 \mathrm{a}^{n+2}\mathrm{S}_{n+1}(\lambda \mathrm{a}).\mathrm{C}$$

$$\mathrm{A}=+\lambda^2 \mathrm{b}^{-n+1}\mathrm{T}_{n-1}(\lambda \mathrm{b}).\mathrm{D}, \quad \mathrm{B}=-\lambda^2 \mathrm{b}^{-n+1}\mathrm{S}_{n-1}(\lambda \mathrm{b}).\mathrm{D}.$$

Introducing a new constant $\mathfrak{A}$, we may therefore put

$$\left.\begin{array}{l} \mathrm{C}=\quad \mathfrak{A}.\mathrm{b}^{-n+1}\mathrm{S}_{n-1}(\lambda \mathrm{b}) \\ \mathrm{A}=-\mathfrak{A}\lambda^2 \mathrm{a}^{n+2}\mathrm{b}^{-n+1}\mathrm{T}_{n+1}(\lambda \mathrm{a})\mathrm{S}_{n-1}(\lambda \mathrm{b}) \\ \mathrm{B}=+\mathfrak{A}\lambda^2 \mathrm{a}^{n+2}\mathrm{b}^{-n+1}\mathrm{S}_{n+1}(\lambda \mathrm{a})\mathrm{S}_{n-1}(\lambda \mathrm{b}) \\ \mathrm{D}=-\mathfrak{A}\mathrm{a}^{n+2}\mathrm{S}_{n+1}(\lambda \mathrm{a}) \end{array}\right\} \quad \ldots\ldots\ldots \quad (61)$$

with the further equation, for the determination of λ,

$$\mathrm{S}_{n+1}(\lambda \mathrm{a}).\mathrm{T}_{n-1}(\lambda \mathrm{b})-\mathrm{T}_{n+1}(\lambda \mathrm{a}).\mathrm{S}_{n-1}(\lambda \mathrm{b})=0 \quad \ldots\ldots\ldots \quad (62)$$

To construct this equation, I observe that

$$\mathrm{S}_n, \equiv x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x},$$

may be written

$$\mathrm{X}_1{}^n \sin\left(x+\frac{n\pi}{2}\right)+\mathrm{X}_2{}^n \sin\left(x+\frac{n+1}{2}\pi\right) \quad \ldots\ldots\ldots \quad (\mathrm{B}_{11})$$

and that

$$\mathrm{T}_n, \text{ which } =x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin\left(\frac{\pi}{2}+x\right)}{x},$$

may in like manner be written

$$\mathrm{X}_1{}^n \sin\left(x+\frac{n+1}{2}\pi\right)+\mathrm{X}_2{}^n \sin\left(x+\frac{n+2}{2}\pi\right) \quad \ldots\ldots\ldots \quad (\mathrm{B}_{12})$$

In fact

$$\left.\begin{array}{l} x.\mathrm{X}_1{}^n=1-\dfrac{(n-1)n(n+1)(n+2)}{2.4}.\dfrac{1}{x^2}+\dfrac{(n-3)(n-2)\ldots(n+3)(n+4)}{2.4.6.8}.\dfrac{1}{x^4}-\&\mathrm{c}. \\ x.\mathrm{X}_2{}^n=\dfrac{n(n+1)}{2}.\dfrac{1}{x}-\dfrac{(n-2)(n-1)\ldots(n+2)(n+3)}{2.4.6}.\dfrac{1}{x^3}+\ldots \end{array}\right\} \quad (\mathrm{B}_{13})$$

With this notation we may put

$$\mathrm{S}_{n+1}(\lambda \mathrm{a})=\mathrm{A}_1^{n+1}\sin\left(\lambda\mathrm{a}+\frac{n+1}{2}\pi\right)+\mathrm{A}_2^{n+1}\cos\left(\lambda\mathrm{a}+\frac{n+1}{2}\pi\right)$$

$$\mathrm{T}_{n+1}(\lambda \mathrm{a})=\mathrm{A}_1^{n+1}\cos\left(\lambda\mathrm{a}+\frac{n+1}{2}\pi\right)-\mathrm{A}_2^{n+1}\sin\left(\lambda\mathrm{a}+\frac{n+1}{2}\pi\right),$$

$$\mathrm{S}_{n-1}(\lambda \mathrm{b})=\mathrm{B}_1^{n-1}\sin\left(\lambda\mathrm{b}+\frac{n-1}{2}\pi\right)+\mathrm{B}_2^{n-1}\cos\left(\lambda\mathrm{b}+\frac{n-1}{2}\pi\right)$$

$$\mathrm{T}_{n-1}(\lambda \mathrm{b})=\mathrm{B}_1^{n-1}\cos\left(\lambda\mathrm{b}+\frac{n-1}{2}\pi\right)-\mathrm{B}_2^{n-1}\sin\left(\lambda\mathrm{b}+\frac{n-1}{2}\pi\right)$$

Substituting in (62),

$$(\mathrm{A}_2^{n+1}\mathrm{B}_1^{n-1}-\mathrm{A}_1^{n+1}\mathrm{B}_2^{n-1})\cos(\lambda(\mathrm{b}-\mathrm{a})-\pi)-(\mathrm{A}_2^{n+1}\mathrm{B}_2^{n-1}+\mathrm{A}_1^{n+1}\mathrm{B}_1^{n-1})\sin(\lambda(\mathrm{b}-\mathrm{a})-\pi)=0$$

or more simply

$$(\mathrm{A}_1^{n+1}\mathrm{B}_1^{n-1}+\mathrm{A}_2^{n+1}\mathrm{B}_2^{n-1})\sin.\lambda(\mathrm{b}-\mathrm{a})-(\mathrm{A}_2^{n+1}\mathrm{B}_1^{n-1}-\mathrm{A}_1^{n+1}\mathrm{B}_2^{n-1})\cos.\lambda(\mathrm{b}-\mathrm{a})=0.$$

This equation may be further simplified; for, putting

$$\tan\vartheta'=\frac{\mathrm{X}_2^n}{\mathrm{X}_1^n},\ \tan\alpha'=\frac{\mathrm{A}_2^{n+1}}{\mathrm{A}_1^{n+1}},\ \tan\beta'=\frac{\mathrm{B}_2^{n-1}}{\mathrm{B}_1^{n-1}}\quad .\ .\ .\ .\ .\ .\quad (63)$$

it becomes

$$\sin\{\lambda.\overline{\mathrm{b}-\mathrm{a}}-\overline{\alpha'-\beta'}\}=0,$$

whence

$$\lambda.\overline{\mathrm{b}-\mathrm{a}}=\alpha'-\beta'+i\pi\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (64)$$

here α' and β' are the smallest positive angles which satisfy the above equations; and ϑ may, when $2x>n(n+1)$, be expanded in ascending powers of $\frac{1}{x}$ by GREGORY'S series.

Up to $\frac{1}{x^3}$,

$$\vartheta=\frac{n(n+1)}{2x}+\frac{(n-1)n(n+1)(n+2)}{2.4}.\frac{1}{x^2}-\frac{n(n+1)}{2.4.6}[3.(n^2+n)^2-8(n^2+n)+12].\frac{1}{x^3}+\ldots\quad (65)$$

The values of α' and β' in terms of λa and λb may be similarly found, and when substituted in equation (64) will give the means of finding all the values of λ. These being found, the complete expression for P may be written down and the values of the constants therein determined from the initial circumstances by known methods; we shall therefore proceed to find the value of P at a point outside the shell.

The potentials at a point outside the shell and in its substance may be written respectively

$$\mathrm{P}=\Sigma \mathrm{D}'\left(\frac{\mathrm{b}}{r}\right)^{n+1}\mathrm{U}_n e^{-\frac{\lambda^2\sigma}{4\pi}t}$$

and

$$\mathrm{P}=\Sigma \mathrm{D}'\lambda^2\mathrm{b}^2(\mathrm{S}_n(x)\mathrm{T}_{n-1}(\beta)-\mathrm{T}_n(x)\mathrm{S}_{n-1}(\beta))e^{-\frac{\lambda^2\sigma}{4\pi}t},$$

wherein, as before, $x=\lambda r$, $\beta=\lambda\mathrm{b}$.

Let us write

$$\mathrm{H}_n=(\mathrm{S}_n\mathrm{T}_{n-1}(\beta)-\mathrm{T}_n\mathrm{S}_{n-1}(\beta))\lambda^2\mathrm{b}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

where $\mathrm{T}_{n-1}(\beta)$ and $\mathrm{S}_{n-1}(\beta)$ are to be treated as simply certain constant multipliers; H_n will satisfy all the formulæ of reduction which we have found to hold separately for S_n and T_n; and we have further these particular values

$$\left.\begin{aligned}&\mathrm{H}_n(\beta)=1,\ \mathrm{H}_{n+1}(\alpha)=0 \text{ by equation (62)}\\ &\text{and by the equation } (\mathrm{B}_9),\ \mathrm{H}_{n+1}(\beta)=-\frac{2n+1}{\beta}\mathrm{H}_n(\beta)=-\frac{2n+1}{\lambda\mathrm{b}}\end{aligned}\right\} \quad . \quad . \quad (67)$$

Now $\mathrm{H}_n(x)$ satisfies the equation

$$\frac{d^2\mathrm{H}}{dr^2}+\frac{2}{r}\frac{d\mathrm{H}}{dr}+\left(\lambda^2-\frac{n(n+1)}{r^2}\right)\mathrm{H}=0,$$

and if H′ correspond to any other value of λ, viz.: λ′, we have also

$$\frac{d^2\mathrm{H}'}{dr^2}+\frac{2}{r}\frac{d\mathrm{H}'}{dr}+\left(\lambda'^2-\frac{n(n+1)}{r^2}\right)\mathrm{H}'=0,$$

whence

$$\left\{r^2\left(\mathrm{H}\frac{d\mathrm{H}'}{dr}-\mathrm{H}'\frac{d\mathrm{H}}{dr}\right)\right\}_{\mathrm{a}}^{\mathrm{b}}+(\lambda'^2-\lambda^2)\int_{\mathrm{a}}^{\mathrm{b}}r^2\mathrm{H}\mathrm{H}'dr=0.$$

But, by equations (B_4),

$$r\frac{d\mathrm{H}_n}{dr}=n\mathrm{H}_n+\lambda r\mathrm{H}_{n+1}$$

$$r\frac{d\mathrm{H}'_n}{dr}=n\mathrm{H}'_n+\lambda' r\mathrm{H}'_{n+1}$$

and, therefore,

$$(\lambda'^2-\lambda^2)\int_{\mathrm{a}}^{\mathrm{b}}r^2\mathrm{H}\mathrm{H}'dr=\left\{r^3(\lambda\mathrm{H}_{n+1}\mathrm{H}'_n-\lambda'\mathrm{H}_n\mathrm{H}'_{n+1})\right\}_{\mathrm{a}}^{\mathrm{b}} \quad . \quad . \quad . \quad . \quad . \quad (68)$$

But if λ' be also one of the roots of equation (62), both H_{n+1} and H'_{n+1} vanish at the lower limit; and at the upper limit

$$H_n = H'_n = 1 \text{ and } \lambda H_{n+1} = \lambda' H'_{n+1} = -\frac{2n+1}{b}$$

hence

$$\int_a^b r^2 H_n H'_n dr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

If we put $\lambda' = \lambda + \delta\lambda$, and then make $\delta\lambda$ infinitely small, we arrive at the value of $\int_a^b r^2 H^2 dr$: for

$$H'_n = H_n(\lambda r + \delta\lambda . r) = H_n + r\delta\lambda \frac{dH_n}{dx}$$

$$= H_n + \frac{1}{\lambda}\delta\lambda(\lambda r H_{n+1} + nH_n) \text{ by } (B_4)$$

and

$$H'_{n+1} = H_{n+1}(\lambda r + \delta\lambda . r) = H_{n+1} + \frac{\delta\lambda}{\lambda}(-\lambda r H_n - \overline{n+1}H_{n+1}) \text{ by } (B_5)$$

Putting in these values in (68)

$$\int_a^b r^2 H_n^2 dr = \frac{1}{2\lambda}\Big[r^3\{\lambda r(H_n^2 + H^2_{n+1}) + 2nH_nH_{n+1}\}\Big]_a^b$$

$$= \frac{b^2}{2\lambda^2}(2n+1) - \frac{\lambda^2 b^2 a^4}{2}(S_n(\lambda a)T_{n-1}(\lambda b) - T_n(\lambda a)S_{n-1}(\lambda b))^2 \qquad (70)$$

Let us call this integral I_n; we may now separate the coefficient D' due to any term of U_n, say $\cos m\phi P_m^n$: for if we put in equation (58) $t=0$, $P = -P_0$, multiply across by $\cos m\phi d\phi . P_m^n \sin\theta d\theta . H_n r^2 dr$ and integrate throughout the body of the shell, all the terms on the right hand side disappear except those which have the some m, n, and λ; and, if we note that

$$\int_0^{2\pi}\cos^2 m\phi d\phi = \pi, \quad \int_0^\pi (P_m^n)^2 \sin\theta d\theta = \frac{2}{2n+1} \cdot \frac{(n+m)!\,(n-m)!}{(1.3.5\ldots\overline{2n-1})^2} = J_m^n \text{ say,}$$

we find

$$D' = -\int_0^{2\pi}\cos m\phi d\phi \int_0^\pi P_m^n . \sin\theta d\theta \int_a^b P_0 H_n r^2 dr \div \pi J_m^n I_n.$$

The coefficient due to $\sin m\phi P_m^n$ may be similarly found.

Putting all the results together we find for the value of P at an outside point

$$\mathrm{P}=-\frac{1}{\pi}\,{}_m\sum_0^\infty\,{}_n\sum_0^\infty\left(\frac{\mathrm{b}}{r}\right)^{n+1}{}_\lambda\Sigma\int_0^{2\pi}\cos m(\phi-\phi')d\phi'\,.\int_0^\pi \mathrm{P}_m{}^n(\theta)\mathrm{P}_m{}^n(\theta')\sin\theta' d\theta'\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{H}_n(\lambda r')\frac{\mathrm{P}'_0 r'^2}{\mathrm{I}_n\mathrm{J}_m{}^n}dr'$$

$$\times e^{-\frac{\lambda^2\sigma}{4\pi}t}\quad .\quad .\quad .\quad .\quad .\quad (71)$$

From this formula the action of the shell on an external point due to any inducing system may be found.

We proceed to inquire what will be the result when the shell is supposed to become infinitely thin.

Infinitely thin shell.

§ 13. Putting $\mathrm{b}-\mathrm{a}=c$, $\sigma=\mathrm{R}c$, the exponential-function $e^{-\lambda^2 c.\frac{\mathrm{R}t}{4\pi}}$, will be finite only for such values of λ as make $\lambda^2 c$ finite: if $\lambda^2 c$ become indefinitely great the corresponding terms will rapidly disappear. More accurately, the terms for which $\lambda^2 c$ is finite will die away infinitely less rapidly than those for which it is infinitely great.

If, in the equation

$$\lambda c=\alpha'-\beta'+i\pi,$$

we take as a first approximation $\lambda=\frac{i\pi}{c}$, we may readily expand λc in a series of the form $i\pi+\mathrm{E}_1.c+\mathrm{E}_2c^2+\ldots$; but in this case $\lambda^2 c=\frac{i^2\pi}{c}$ approximately when c is very great.

The corresponding system of currents will rapidly decay. It is therefore by putting $i=0$ that we obtain the currents which have the greatest persistence. The equation in λ is

$$\lambda^2 c=\frac{(n+2)(n+1)}{2\mathrm{a}}-\frac{(n-1)n}{2\mathrm{b}}+\text{ terms in }\lambda^{-1},\ \lambda^{-2},\ \ldots,$$

and, for a first approximation, b=a and

$$\lambda^2 c=\frac{1}{\mathrm{a}}(2n+1)\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad (72)$$

Hence, by equations (58) and

$$\left.\begin{array}{rl}\text{for points outside the shell,} & \mathrm{P}=\Sigma\mathfrak{C}\left(\frac{\mathrm{a}}{r}\right)^{n+1}\mathrm{U}_n e^{-\frac{\mathrm{R}}{4\pi\mathrm{a}}(2n+1)t}\\ \text{,, \qquad within the shell,} & \mathrm{P}=\Sigma\mathfrak{C}\left(\frac{r}{\mathrm{a}}\right)^{n}\mathrm{U}_n e^{-\frac{\mathrm{R}}{4\pi\mathrm{a}}(2n+1)t}\end{array}\right\}\quad .\quad .\quad .\quad (73)$$

To interpret these results :—

(1.) For an external point,

$$\left.\begin{array}{l}\text{we observe that when } t=0,\ \mathrm{P}=\Sigma\mathfrak{C}\left(\frac{\mathrm{a}}{r}\right)^{n+1}\mathrm{U}_n\\ \text{and that, always, } \mathrm{P}=e^{-\frac{\mathrm{R}}{4\pi\mathrm{a}}t}.\Sigma\mathfrak{C}\left(\frac{\mathrm{a}'}{r}\right)^{n+1}\mathrm{U}_n\\ \text{where } \mathrm{a}'=\mathrm{a}e^{-\frac{\mathrm{R}}{2\pi\mathrm{a}}.t}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

The result, therefore, of an impulse on the sheet is such that initially the currents exert the same action on an external point, as a positive image of the magnetic system placed at the position of the electric image at the surface of the sheet.

The points in which this imaginary magnetism is distributed then move towards the centre according to the law $\rho'=\rho e^{-\frac{\mathrm{R}t}{2\pi\mathrm{a}}}$, while at the same time the intensity at each point increases according to the law $\mathrm{I}'=\mathrm{I}e^{\frac{\mathrm{R}t}{4\pi\mathrm{a}}}$.

When $\mathrm{a}=\infty$ we may take $\rho=\mathrm{a}$ and $\rho-\rho'=\frac{\mathrm{R}}{2\pi}t$, and I' is constant; this result reproduces MAXWELL's expressions for a plane sheet.

(2.) For a point on the other side of the shell

$$\left.\begin{array}{l}\mathrm{P}=e^{-\frac{\mathrm{R}t}{4\pi\mathrm{a}}}\Sigma.\mathfrak{C}\left(\frac{r}{\mathrm{a}''}\right)^{n}\mathrm{U}_n\\ \text{where}\\ \mathrm{a}''=\mathrm{a}e^{+\frac{\mathrm{R}t}{2\pi\mathrm{a}}}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

The effect is, therefore, the same as if the inducing magnetic system were reversed in sign, and the points in which it is distributed were to move off to infinity in lines passing through the centre of the shell according to the law

$$\rho'=\rho e^{\frac{\mathrm{R}t}{2\pi\mathrm{a}}},$$

the intensity at each point diminishing according to the law

$$\mathrm{I}'=\mathrm{I}e^{-\frac{\mathrm{R}t}{4\pi\mathrm{a}}}.$$

§ 14. This result is so interesting that it may be well to give an independent demonstration of it. Employing, as hitherto, spherical coordinates, let Φ be the current function for the currents in the sheet; P, the potential due to imaginary matter distributed over it, with surface density Φ; the magnetic and vector potentials of the current system may be written

$$\Omega=-\frac{1}{\mathrm{a}}\frac{d}{dr}(\mathrm{P}r),\ \ \mathrm{F}=0,\ \ \mathrm{G}=+\frac{1}{\mathrm{a}}\frac{d\mathrm{P}}{\sin\theta d\phi},\ \ \mathrm{H}=-\frac{d\mathrm{P}}{\mathrm{a}d\theta}.$$

The components of the current are

$$u=0,\ v=-\frac{d\Phi}{\sin\theta d\phi},\ w=\frac{d\Phi}{d\theta}.$$

The equations, therefore, of the current are

$$-\frac{\mathrm{R}}{4\pi}\frac{d\Phi}{\sin\theta d\phi}=-\frac{d^2\mathrm{P}}{\mathrm{a}\sin\theta d\phi dt}-\frac{d\psi}{\mathrm{a}d\theta}$$

$$+\frac{\mathrm{R}}{4\pi}\frac{d\Phi}{d\theta}=+\frac{d^2\mathrm{P}}{\mathrm{a}d\theta dt}-\frac{d\psi}{\mathrm{a}\sin\theta d\phi}.$$

which is satisfied by

$$\psi=0,\ \Phi=\frac{1}{\mathrm{a}}\frac{d\mathrm{P}}{dt}.$$

Outside the sheet and within its inner surface,

$$\mathrm{F}=0,\ \mathrm{G}=\frac{d\mathrm{P}}{\sin\theta d\phi},\ \mathrm{H}=-\frac{d\mathrm{P}}{d\theta},$$

$$\nabla^2\mathrm{P}=0.$$

Hence

outside $\mathrm{P}_0=\Sigma\mathrm{A}\left(\frac{\mathrm{a}}{r}\right)^{n+1}\mathrm{U}_n e^{-\lambda^2\frac{\mathrm{R}}{4\pi}t}$

inside $\mathrm{P}_i=\Sigma\mathrm{A}\left(\frac{r}{\mathrm{a}}\right)^{n}\mathrm{U}_n e^{-\lambda^2\frac{\mathrm{R}}{4\pi}t}$;

at the surface $\Phi=-\frac{1}{\mathrm{a}}\Sigma\mathrm{A}\mathrm{U}_n e^{-\lambda^2\frac{\mathrm{R}}{4\pi}t}.\frac{\lambda^2\mathrm{R}}{4\pi}.$

But we have also the further condition

$$\left(\frac{d\mathrm{P}_0}{dr}-\frac{d\mathrm{P}_i}{dr}\right)_{r=a}=4\pi\Phi,$$

hence

$$\lambda^2=\frac{1}{\mathrm{a}}(2n+1);$$

and therefore

$$\mathrm{P}_0=\Sigma\mathrm{A}\left(\frac{\mathrm{a}}{r}\right)^{n+1}\mathrm{U}_n e^{-(2n+1)\frac{\mathrm{R}}{4\pi\mathrm{a}}t},$$

$$\mathrm{P}_i=\Sigma\mathrm{A}\left(\frac{r}{\mathrm{a}}\right)^{n}\mathrm{U}_n e^{-(2n+1)\frac{\mathrm{R}}{4\pi\mathrm{a}}t}.$$

This corresponds with the result obtained from the more general investigation of last article.

Moving conductor.

§ 15. When the conductor is moving in any manner, the parts of the electromotive force due to the motion are, in the directions of the three axes, respectively equal to

$$\gamma\dot{y}-\beta\dot{z},\ \alpha\dot{z}-\gamma\dot{x},\ \beta\dot{x}-\alpha\dot{y}.$$

The form of these expressions shows that, when it is desirable to resolve the electromotive force in any other directions at right angles, the components will be

$$\gamma\mathrm{V}-\beta\mathrm{W},\ \alpha\mathrm{W}-\gamma\mathrm{U},\ \beta\mathrm{U}-\alpha\mathrm{V}\ .\ .\ .\ .\ .\ .\ .\ .\ \quad (1)$$

α, β, γ being the components of magnetic force, and U, V, W the components of velocity in these directions. When the motion of the body is uniform, as when it is revolving uniformly round an axis, and when it is symmetrical about this axis, the electric state may after a certain time be supposed to have become constant; there is then no variation of electric displacement at each point of space, and the currents of conduction become the total currents. We have then

$$u=\sigma\mathrm{P}=\frac{\sigma}{\mathrm{K}}\cdot f,\ .\ .$$

and therefore

$$\epsilon=\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \quad (2)$$

There is thus no free electricity in the substance of the conductor, though there may be electric potential: and the normal component of the current across the surface of the body is zero: that is to say,

$$lu+mv+nw=0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \quad (3)$$

It is to be noted here that we are here dealing with the state of definite points of space: these are invariable. The different parts of the conductor take different conditions as they move from one point of space to another.

Since the currents are confined to the conductor the vector potential due to them and its differential coefficients will all be continuous on passing across the surface of the conductor.

With these preliminary observations we proceed to the consideration of a solid of infinite extent and thickness bounded by a plane face, revolving round an axis normal to its face.

Infinite solid bounded by plane face.

§ 16. Let the axis of z be chosen to coincide with the axis of revolution and the origin on the face, and revolve along $d\rho$, $\rho d\phi$, dz;

$$\mathrm{U}=0,\ \mathrm{V}=\omega\rho,\ \mathrm{W}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

ω being the angular velocity supposed uniform.

The components of electromotive force are

$$\left.\begin{aligned}\mathrm{P}&=\ \ \omega\rho\gamma-\frac{d\psi}{d\rho}\\ \mathrm{Q}&=\qquad\ -\frac{d\psi}{\rho d\phi}\\ \mathrm{R}&=-\omega\rho\alpha-\frac{d\psi}{dz}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The external magnetic force may be represented by

$$\Omega_0=\frac{d\mathrm{P}_0}{dz},\ \mathrm{F}_0=-\frac{d\mathrm{P}_0}{\rho d\phi},\ \mathrm{G}_0=\frac{d\mathrm{P}_0}{d\rho},\ \mathrm{H}_0=0,$$

and all the conditions of the problem are satisfied by taking for the vector potential of the currents in the solid

$$\mathrm{F}=-\frac{d\mathrm{P}}{\rho d\phi},\ \mathrm{G}=\frac{d\mathrm{P}}{d\rho},\ \mathrm{H}=0.$$

The total magnetic force is the sum of the parts due to the currents in the sheet and the external force: it is therefore given by

$$\alpha=-\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\rho dz},\ \beta=-\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{\rho d\phi dz},\ \gamma=\frac{1}{\rho}\frac{d}{d\rho}\cdot\left(\rho\frac{d(\mathrm{P}+\mathrm{P}_0)}{d\rho}\right)+\frac{1}{\rho^2}\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\phi^2}.$$

The currents may be denoted by

$$u=-\frac{d\Phi}{\rho d\phi},\ v=\frac{d\Phi}{d\rho},\ w=0,$$

$$4\pi\Phi=-\nabla^2\mathrm{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6_1)$$

If we now substitute for P, Q, R their equivalents σu, σv, 0, the second of equations (5) becomes

$$\sigma\frac{d\Phi}{d\rho}=-\frac{d\psi}{\rho d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6_2)$$

which is satisfied by

$$\Psi=\rho\alpha\frac{d\chi}{d\rho},\ \Phi=-\frac{d\chi}{d\phi}=-\frac{1}{4\pi}\nabla^2\mathrm{P} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (6_3)$$

and from the last equation

$$\omega\rho\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\rho dz}-\sigma\rho\frac{d^2\chi}{d\rho dz}=0,$$

whence

$$\sigma\chi=\omega(\mathrm{P}+\mathrm{P}_0) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (7)$$

By equation (6_1) this may be written

$$\frac{\sigma}{4\pi}\nabla^2\mathrm{P}=\omega\frac{d}{d\phi}(\mathrm{P}+\mathrm{P}_0) \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

This is the characteristic equation of steady currents in a rotating conductor: we may show that the first of equations (5) is also satisfied, for this requires that

$$-\frac{\sigma}{\rho}\cdot\frac{d\Phi}{d\phi}=\omega\frac{d}{d\rho}\cdot\rho\frac{d(\mathrm{P}+\mathrm{P}_0)}{d\rho}+\frac{\omega}{\rho}\frac{d^2(\mathrm{P}+\mathrm{P}_0)}{d\phi^2}-\omega\frac{d}{d\rho}\cdot\rho\frac{d}{d\rho}(\mathrm{P}+\mathrm{P}_0),$$

or

$$-\sigma\Phi=\omega\frac{d}{d\phi}(\mathrm{P}+\mathrm{P}_0),$$

which corresponds with the former equations.

All the conditions of the problem will be satisfied by determining P subject to the following conditions :—

(1.) It must satisfy equation (8) within the solid, and vanish when $z=-\infty$.

(2.) „ „ $\nabla^2\mathrm{P}=0$ outside the solid, and vanish when $z=+\infty$.

(3.) P and $\frac{d\mathrm{P}}{dz}$ must be the same outside and inside the solid when $z=0$.

Let us now suppose that P_0 can be expanded within the solid in a series of the form $\mathrm{P}_0=\Sigma(\mathrm{A}e^{im\phi}+\mathrm{A}'e^{-im\phi})\mathrm{J}_m(\kappa\rho)e^{\kappa z}$: each of these terms verifying the equation $\nabla^2\mathrm{P}_0=0$ as they ought to do, and vanishing when $z=-\infty$. We shall show presently how this may be done.

The corresponding part of P due to $\mathrm{A}e^{im\phi}$ is

$$\mathrm{P}=-\mathrm{A}e^{im\phi}\mathrm{J}_m(\kappa\rho)e^{\kappa z}+\Pi \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

where

$$\frac{\sigma}{4\pi}\nabla^2\Pi=+\omega\frac{d\Pi}{d\phi}.$$

Putting therefore

$$\Pi=\mathrm{B}e^{mi\phi}\mathrm{J}_m(\kappa\rho)e^{\mu z} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (10)$$

we have

$$-\kappa^2+\mu^2=i.\frac{4\pi m\omega}{\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\therefore\ \mu=\pm\sqrt{\kappa^2-i\frac{4\pi m\omega}{\sigma}}=\pm(p-iq) \text{ suppose.}$$

We must take only the positive sign since $\Pi=0$ when $z=-\infty$.

Thus

$$P=e^{im\phi}J_m(\kappa\rho)(-Ae^{\kappa z}+Be^{\mu z}) \text{ within the solid ;}$$

outside

$$P=e^{im\phi}J_m(\kappa\rho).Ce^{-\kappa z},$$

and the conditions of continuity give us

$$-A+B=C,\ -\kappa A+\mu B=-\kappa C,$$

whence

$$B=\frac{2\kappa}{\kappa+\mu}.A,\ C=\frac{\kappa-\mu}{\kappa+\mu}.A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Let us confine our attention now to the value of C which expresses the action of the sheet on external points

$$C=A.\frac{\kappa-p+qi}{\kappa+p-qi}=A(p'+q'i).$$

similarly, corresponding to the term $A'e^{-im\phi}$, we find

$$C'=A'(p'-q'.i).$$

Let

$$Ae^{im\phi}+A'e^{-im\phi}=\mathfrak{A}\cos m\phi+\mathfrak{B}\sin m\phi$$
$$Ce^{im\phi}+C'e^{-im\phi}=\mathfrak{C}\cos m\phi+\mathfrak{D}\sin m\phi$$

$$\left.\begin{aligned}\mathfrak{C}&=C+C'=\mathfrak{A}p'+\mathfrak{B}q'\\ \mathfrak{D}&=(C-C')i=\mathfrak{B}p'-\mathfrak{A}q'\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Let

$$p'+q'i=M'.e^{i\vartheta},$$

then corresponding to a term in

$$P_0=(\mathfrak{A}\cos m\phi+\mathfrak{B}\sin m\phi)e^{\kappa z}J_m(\kappa\rho),$$

there exists in P (outside) the term

$$[\mathfrak{A}M'\cos(m\phi+\vartheta')+\mathfrak{B}M'\sin(m\phi+\vartheta')]e^{-\kappa z}J_m(\kappa\rho) \quad . \quad . \quad . \quad . \quad (14)$$

the term is therefore altered in intensity in the ratio M′ : 1 and the azimuth increased by the angle ϑ'. If we put

$$\mu^2 = \mathrm{M}^2 e^{-2i\vartheta},$$

we can easily prove that

$$\mathrm{M}' = \frac{q}{p+q} = \frac{\sin\vartheta}{\sin\vartheta + \cos\vartheta},\ \vartheta' = \frac{\pi}{2} + \tan^{-1}\left(\frac{\mathrm{M}\sin\vartheta}{\kappa}\right)$$

where also

$$\tan 2\vartheta = \frac{4\pi m\omega}{\sigma},\ \mathrm{M}^4 = \kappa^4 + \left(\frac{4\pi m\omega}{\sigma}\right)^2.$$

The action of the solid will therefore be the reverse of the original magnetisation and M′ is a proper fraction.

The coefficients $\mathfrak{A}$, $\mathfrak{B}$ are found by considering the value of P_0 when $z=0$; call it Z_0. Then as we have found

$$\mathrm{Z}_0 = \frac{1}{\pi}\Sigma\int_0^{2\pi}\cos m(\phi-\phi')d\phi'\int_0^\infty \kappa d\kappa \mathrm{J}_m(\kappa\rho)\int_0^\infty \mathrm{Z}'_0 \mathrm{J}_m(\kappa\rho')\rho' d\rho' . \quad . \quad . \quad . \qquad (15)$$

Plate of finite or infinitesimal thickness.

§ 17. For the case of a plate bounded by two parallel planes at distance b apart, we may satisfy all the conditions by taking the same general forms for the vector potentials and for the currents in the sheet: and the characteristic equation for the determination of P will remain the same as before. If we suppose the inducing magnetism distributed on the positive side of the plate, we may express P_0 within the plate as a series of terms of the form

$$\mathrm{A}e^{im\phi}\mathrm{J}_m(\kappa\rho)e^{\kappa z},$$

the origin being in the axis of revolution, and on the positive surface of the plate. The forms for the vector potential due to the currents are all given by taking for P a series of terms of the type

i. In the substance of plate $-\mathrm{A}e^{im\phi}\mathrm{J}_m(\kappa\rho)e^{\kappa z} + e^{im\phi}\mathrm{J}_m(\kappa\rho)(\mathrm{B}_1 e^{\mu z} + \mathrm{B}_2 e^{-\mu z})$
ii. Outside the plate, z positive, $e^{im\phi}\mathrm{J}_m(\kappa\rho)\mathrm{C}e^{-\kappa z}$
iii. Outside, z negative, $e^{im\phi}\mathrm{J}_m(\kappa\rho).\mathrm{D}e^{+\kappa z}$

where

$$\mu = \sqrt{\kappa^2 - i\frac{4\pi\omega m}{\sigma}}.$$

On the two faces of the plate P and $\frac{dP}{dz}$ are to change continuously on crossing the faces.

The final result of the calculation gives

$$C=A.\frac{(\kappa^2-\mu^2)(e^{\mu b}-e^{-\mu b})}{(\kappa+\mu)^2e^{\mu b}-(\kappa-\mu)^2e^{-\mu b}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

when b is indefinitely small, we treat $R=\frac{\sigma}{b}$ as finite, and therefore while κ is finite μ is infinitely great, but μ^2b is finite and equal to $-\frac{4\pi m\omega}{R}.i.$ To find C, we have then

$$\frac{C}{C+A}=\frac{\kappa^2-\mu^2}{2\kappa}.\frac{e^{\mu b}-e^{-\mu b}}{\kappa(e^{\mu b}-e^{-\mu b})+\mu(e^{\mu b}+e^{-\mu b})}$$

$$=\frac{(\kappa^2-\mu^2)b}{2\kappa}.\frac{1+\frac{\mu^2b^2}{6}+\ldots}{1+\kappa b+\frac{\mu^2b^2}{2}+\frac{\kappa b.\mu^2b^2}{6}+\ldots}$$

$$=\frac{2\pi\omega m}{\kappa R}.i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If Q be the value (due to a positive image of P_0) on the positive side of the plate, and P the value of P due to the currents,

$$Q=\Sigma Ae^{im\phi}J_m(\kappa\rho)e^{+\kappa z},\ P=\Sigma Ce^{im\phi}J_m(\kappa\rho)e^{-\kappa z}$$

then, putting

$$\varpi=\frac{2\pi\omega}{R}$$

$$\frac{dP}{dz}-\varpi\frac{dP}{d\phi}=\varpi\frac{dQ}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This corresponds with MAXWELL'S result; see also equation (34).

Spherical shell and sphere.

§ 18. We shall now treat the case of a spherical shell, whose outer and inner radii we shall take to be b, a.

The expressions for the electromotive force are (taking dx, dy, dz to correspond respectively with dr, $rd\theta$, $r\sin\theta d\phi$).

$$\left.\begin{aligned} \mathrm{P} &= -\omega r \sin\theta . \beta - \frac{d\psi}{dr} \\ \mathrm{Q} &= +\omega r \sin\theta . \alpha - \frac{d\psi}{r d\theta} \\ \mathrm{R} &= \qquad\qquad\quad - \frac{d\psi}{r \sin\theta d\phi} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

Since, with the present notation, we have $\mathrm{U}=0$, $\mathrm{V}=0$, $\mathrm{W}=\omega r \sin\theta$, we shall choose the same forms for the vector potentials and current-functions as in § 8, viz. :—

$$\Omega_0 = \frac{d}{dr}(\mathrm{P}_0 r),\ \mathrm{F}_0=0,\ \mathrm{G}_0 = -\frac{d\mathrm{P}_0}{\sin\theta d\phi},\ \mathrm{H}_0 = \frac{d\mathrm{P}_0}{d\theta},$$

$$\mathrm{F}=0,\ \mathrm{G} = -\frac{d\mathrm{P}}{\sin\theta d\phi},\ \mathrm{H} = \frac{d\mathrm{P}}{d\theta}.$$

The magnetic force α, β, γ is due to both potentials, and thus (see § 3 B, *c*),

$$\alpha = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d\overline{\mathrm{P}+\mathrm{P}_0}}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2\overline{\mathrm{P}+\mathrm{P}_0}}{d\phi^2}\right],\ \beta = -\frac{1}{r}\frac{d^2.\overline{\mathrm{P}+\mathrm{P}_0}.r}{dr d\theta},\ \gamma = -\frac{1}{r\sin\theta}\frac{d^2.\overline{\mathrm{P}+\mathrm{P}_0}.r}{dr d\phi}.$$

The currents will be denoted by

$$u=0,\ v = -\frac{d\Phi}{\sin\theta d\phi},\ w = \frac{d\Phi}{d\theta},$$

where

$$4\pi\Phi = -\nabla^2\mathrm{P} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

observing that

$$\nabla^2\mathrm{P}_0 = 0.$$

Substituting for P, Q, R in terms of u, v, w by OHMS' Law, the last of equations (19) becomes

$$\sigma\frac{d\Phi}{d\theta} = -\frac{d\psi}{r\sin\theta d\phi} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21_1)$$

This is satisfied by taking

$$\Phi = -\frac{d\chi}{d\phi},\ \psi = \sigma . r \sin\theta\frac{d\chi}{d\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21_2)$$

and the first of equations (19) becomes

$$\omega\sin\theta\frac{d^2\overline{\mathrm{P}+\mathrm{P}_0}.r}{dr d\theta} - \sigma\sin\theta\frac{d^2.\chi r}{dr d\theta} = 0,$$

or

$$\sigma\chi = \omega.(\mathrm{P}+\mathrm{P}_0) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21_3)$$

Equation (20) now becomes

$$\frac{\sigma\nabla^2 \mathrm{P}}{4\pi}=\omega\frac{d(\mathrm{P}+\mathrm{P}_0)}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

the same characteristic equation as in § 16.

We may now show that these results verify the second of (19) as the reader may easily convince himself by substitution.

(*a.*) When the inducing magnetism or currents are outside the shell altogether, P_0 will consist of a series of terms of the form

$$\mathrm{A}e^{im\phi}\mathrm{P}_m{}^n.\left(\frac{r}{\mathrm{b}}\right)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and for the value of P we shall have

$$\left.\begin{array}{ll} \text{i. in substance of shell } (\mathrm{b}>r>\mathrm{a}), & \mathrm{P}=\left[-\mathrm{A}\left(\frac{r}{\mathrm{b}}\right)^n+\mathrm{B}_1\mathrm{S}_n(\lambda r)+\mathrm{B}_2\mathrm{T}_n(\lambda r)\right]e^{im\phi}\mathrm{P}_m{}^n \\ \text{ii. outside shell, } r>\mathrm{b} & \mathrm{P}=\mathrm{C}.\left(\frac{\mathrm{b}}{r}\right)^{n+1}e^{im\phi}\mathrm{P}_m{}^n \\ \text{iii. within shell, } r<\mathrm{a} & \mathrm{P}=\mathrm{D}.\left(\frac{r}{\mathrm{b}}\right)^n e^{im\phi}\mathrm{P}_m{}^n \end{array}\right\} \quad . \quad (24)$$

and where

$$\lambda^2=-i.\frac{4\pi m\omega}{\sigma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

we shall put as before

$$x=\lambda r,\ \alpha=\lambda\mathrm{a},\ \beta=\lambda\mathrm{b},\ \mathrm{S}_n(\alpha)=\mathrm{S},\ \mathrm{S}_n(\beta)=\mathrm{S}'_n\ldots$$

When $r=\mathrm{a}$,

$$-\mathrm{A}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^n+\mathrm{B}_1\mathrm{S}_n+\mathrm{B}_2\mathrm{T}_n=\mathrm{D}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^n,$$

$$-\frac{n}{\lambda}.\mathrm{A}\frac{\mathrm{a}^{n-1}}{\mathrm{b}^n}+\mathrm{B}_1\frac{d\mathrm{S}_n}{d\alpha}+\mathrm{B}_2\frac{d\mathrm{T}_n}{d\alpha}=\frac{n\mathrm{D}}{\lambda}.\frac{\mathrm{a}^{n-1}}{\mathrm{b}^n},$$

and when $r=\mathrm{b}$,

$$-\mathrm{A}+\mathrm{B}_1\mathrm{S}'_n+\mathrm{B}_2\mathrm{T}'_n=\mathrm{C},$$

$$-\frac{n}{\lambda\mathrm{b}}\mathrm{A}+\mathrm{B}_1\frac{d\mathrm{S}'n}{d\beta}+\mathrm{B}_2\frac{d\mathrm{T}'_n}{d\beta}=-(n+1)\frac{\mathrm{C}}{\lambda\mathrm{b}}.$$

By the help of the formulæ of reduction in § 10, we can put the result of the calculation of the new coefficients B_1, B_2, C, D into somewhat elegant forms.

Eliminating C and D we find

$$\mathrm{B}_1\mathrm{S}_{n+1}+\mathrm{B}_2\mathrm{T}_{n+1}=0$$

$$\mathrm{B}_1\mathrm{S}'_{n-1}+\mathrm{B}_2\mathrm{T}'_{n-1}=-\frac{2n+1}{\beta},$$

whence we find for C, which is what we chiefly want,

$$\mathrm{C}=\mathrm{A}\left\{-1-\frac{2n+1}{\beta}\cdot\frac{\mathrm{S}_{n+1}\mathrm{T}'_n-\mathrm{T}_{n+1}\mathrm{S}'_n}{\mathrm{S}_{n+1}\mathrm{T}'_{n-1}-\mathrm{T}_{n+1}\mathrm{S}'_{n-1}}\right\}=\mathrm{A}.\frac{\mathrm{S}_{n+1}\mathrm{T}'_{n+1}-\mathrm{T}_{n+1}\mathrm{S}'_{n+1}}{\mathrm{S}_{n+1}\mathrm{T}'_{n-1}-\mathrm{T}_{n+1}\mathrm{S}'_{n-1}}\quad . \quad . \quad . \quad (26)$$

(*b*.) When the inducing magnetism lies partly within the shell (not in its substance) we may take for P_0 a series of the form

$$\mathrm{A}'e^{im\phi}\mathrm{P}_m{}^n\left(\frac{\mathrm{b}}{r}\right)^{n+1};$$

the other expressions for P will retain their forms except that we must take the first term $-\mathrm{A}'\left(\frac{\mathrm{b}}{r}\right)^{n+1}$ instead of $-\mathrm{A}\left(\frac{r}{\mathrm{b}}\right)^n$. The calculation of C is similar to that given above; and, when the algebraic work is performed, we obtain finally

$$\mathrm{C}=\mathrm{A}'\left\{-1-\frac{2n+1}{\lambda\mathrm{a}}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^{n+1}\frac{\mathrm{S}'_n\mathrm{T}'_{n-1}-\mathrm{T}'_n\mathrm{S}'_{n-1}}{\mathrm{S}_{n+1}\mathrm{T}'_{n-1}-\mathrm{T}_{n+1}\mathrm{S}'_{n-1}}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

There are two particular cases of these formulæ which possess a special interest.

(I.) In the case of a *solid sphere* $\mathrm{a}=0$, $\mathrm{S}_{n+1}(\lambda\mathrm{a})=0$.

The inducing magnetism can only belong to type (*a*) and may be represented by $\mathrm{P}_0=\Sigma\mathrm{A}e^{im\phi}\mathrm{P}_m{}^n\left(\frac{r}{\mathrm{b}}\right)^n$: the external effect of the currents is then given, according to (26), by

$$\mathrm{P}=\Sigma\mathrm{A}.\frac{\mathrm{S}_{n+1}(\lambda\mathrm{b})}{\mathrm{S}_{n-1}(\lambda\mathrm{b})}e^{im\phi}\mathrm{P}_m{}^n.\left(\frac{\mathrm{b}}{r}\right)^{n+1}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and it is easy to deduce from this expression the value of P when P_0 is given by terms which are real in ϕ.

(II.) In the case of an *infinitely thin shell,* we must put $\mathrm{b}=\mathrm{a}+c$.

This case is interesting as it constitutes an extension of MAXWELL's result for a plane plate. A glance at the preceding formulæ will show that we have here to expand $\mathrm{S}_\nu\,(\lambda\mathrm{a}+\lambda c)$, $\mathrm{T}_\nu(\lambda\mathrm{a}+\lambda c)$, where $\nu=n-1$, n or $n+1$, in ascending powers of c. Some caution is requisite, however, in doing so on account of the value of λ: for

$\lambda^2 = -\frac{4\pi m w}{\sigma}.i$, and if we put $\sigma = \mathrm{R}c$, and treat R as finite, we must suppose $\lambda^2 c$ finite, and therefore λc is a small quantity of the order $c^{\frac{1}{2}}$, while $\frac{1}{\lambda}$ is also of the order $c^{\frac{1}{2}}$.

The expression

$$\mathrm{S}_{n+1}\mathrm{T}'_{\nu} - \mathrm{T}_{n+1}\mathrm{S}'_{\nu}$$

$$= \mathrm{S}_{n+1}\mathrm{T}_{\nu} - \mathrm{T}_{n+1}\mathrm{S}_{\nu} + \lambda c\left(\mathrm{S}_{n+1}\frac{\mathrm{T}_{\nu}}{d\alpha} - \mathrm{T}_{n+1}\frac{d\mathrm{S}_{\nu}}{d\alpha}\right) + \tfrac{1}{2}\lambda^2 c^2\left(\mathrm{S}_{n+1}\frac{d^2\mathrm{T}_{\nu}}{d\alpha^2} - \mathrm{T}_{n+1}\frac{d^2\mathrm{S}_{\nu}}{d\alpha^2}\right) + \ldots$$

If we write

$$\mathrm{S}_{n+1}\frac{d^p\mathrm{T}_{\nu}}{d\alpha^p} - \mathrm{T}_{n+1}\frac{d^p\mathrm{S}_{\nu}}{d\alpha^p} = u_p,$$

we can readily find a formula of reduction in u_p.

S_{ν} and T_{ν} both satisfy the equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}\{x^2 - \nu(\nu+1)\}y = 0.$$

By LEIBNITZ' theorem

$$x^2y^{(p+2)} + 2(p+1)xy^{(p+1)} + (x^2 + (p-\nu)(p+\nu+1))y^{(p)} + 2py^{(p-1)} + p(p-1)y^{(p-2)} = 0,$$

wherein

$$y^{(p)} = \frac{d^py}{dx^p};$$

whence

$$u_{p+2} + \frac{2(p+1)}{x}u_{p+1} + \left(1 + \frac{(p-\nu)(p+\nu+1)}{x^2}\right)u_p + \frac{2p}{x^2}u_{p-1} + \frac{p(p-1)}{x^2}u_{p-2} = 0,$$

in which we have to write α for x in the problem before us. We conclude that u_{p+2} will usually be of the same degree in $\frac{1}{\alpha}$ as u_p, and when u_0 and u_1 are given we can readily find the remaining values of u_p. But as each of them is multiplied by ever increasing powers of λc, it will be only the first one or two terms which will give finite terms when we treat $\frac{c}{\mathrm{a}}$ as infinitely small. Let us now take up cases (a) and (b) of last article.

(a.) From formula (26) and remembering (B_6), we obtain

$$\frac{\mathrm{C}}{\mathrm{C}+\mathrm{A}} = -\frac{\beta}{2n+1}\cdot\frac{\mathrm{S}_{n+1}\mathrm{T}'_{n+1} - \mathrm{T}_{n+1}\mathrm{S}'_{n+1}}{\mathrm{S}_{n+1}\mathrm{T}'_{n} - \mathrm{T}_{n+1}\mathrm{S}'_{n}};$$

now

$$\mathrm{S}_{n+1}\mathrm{T}'_{n+1}-\mathrm{T}_{n+1}\mathrm{S}'_{n+1}=\lambda c.\left(\mathrm{S}_{n+1}\frac{d\mathrm{T}_{n+1}}{d\alpha}-\mathrm{T}_{n+1}\frac{d\mathrm{S}_{n+1}}{d\alpha}\right)+\ldots$$

$$=\lambda c.-\frac{1}{\lambda^2\mathrm{a}^2}+\text{terms negligible.}$$

$$\mathrm{S}_{n+1}\mathrm{T}'_{n}-\mathrm{T}_{n+1}\mathrm{S}'_{n}=\mathrm{S}_{n+1}\mathrm{T}_{n}-\mathrm{T}_{n+1}\mathrm{S}_{n}+\lambda c\left(\mathrm{S}_{n+1}\frac{d\mathrm{T}_{n}}{d\alpha}-\mathrm{T}_{n+1}\frac{d\mathrm{S}_{n}}{d\alpha}\right)+\ldots$$

$$=\frac{1}{\lambda^2\mathrm{a}^2}+\ldots;$$

and it may be easily shown, by working out the different coefficients of c, c^2, . . . , that the terms neglected are really negligible.

Finally, we obtain

$$\frac{\mathrm{C}}{\mathrm{C}+\mathrm{A}}=\frac{\lambda^2\mathrm{a}c}{2n+1},\text{ putting a for } b \text{ in } \beta.$$

$$=-\frac{4\pi m\omega\mathrm{a}}{(2n+1)\mathrm{R}}.i \quad \ldots\ldots\ldots\ldots \quad (29)$$

(*b*.) Here

$$\frac{\mathrm{C}+\mathrm{A}'}{\mathrm{A}'}=-\frac{2n+1}{\lambda\mathrm{a}}\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^{n+1}\cdot\frac{\mathrm{S}'_n\mathrm{T}'_{n-1}-\mathrm{T}'_n\mathrm{S}'_{n-1}}{\mathrm{S}_{n+1}\mathrm{T}'_{n-1}-\mathrm{T}_{n+1}\mathrm{S}'_{n-1}}.$$

The numerator is, as we know, $\frac{1}{\lambda^2\mathrm{b}^2}$; and the denominator

$$=\mathrm{S}_{n+1}\mathrm{T}_{n-1}-\mathrm{T}_{n+1}\mathrm{S}_{n-1}+\lambda c\left(\mathrm{S}_{n+1}\frac{d\mathrm{T}_{n-1}}{d\alpha}-\mathrm{T}_{n+1}\frac{d\mathrm{S}_{n-1}}{d\alpha}\right)+\ldots$$

the former of which set of terms $=-\frac{2n+1}{\alpha^3}$.

To find the latter we observe that $\frac{d\mathrm{S}_{n-1}}{d\alpha}=\overline{n-1}\frac{\mathrm{S}_{n-1}}{\alpha}+\mathrm{S}_n$, and similarly for T; hence the term in λc is

$$\lambda c\left[-\frac{(n-1)(2n+1)}{\alpha^4}+\frac{1}{\alpha^2}\right],$$

the first term of which is negligible compared to the second: putting these different results together, and making

$$\left(\frac{\mathrm{b}}{\mathrm{a}}\right)^{n+1}=1 \text{ and } \beta=\alpha,$$

$$\frac{\mathrm{C}+\mathrm{A}'}{\mathrm{A}'}=-\frac{2n+1}{\alpha}\cdot\frac{\frac{1}{\alpha^2}}{-\frac{2n+1}{\alpha^3}+\frac{\lambda c}{\alpha^2}}=\frac{1}{1-\frac{\lambda^2\mathrm{a}c}{2n+1}};$$

and hence

$$\frac{\mathrm{C}}{\mathrm{C}+\mathrm{A}'}=\frac{\lambda^2\mathrm{a}c}{2n+1}$$

$$=-\frac{4\pi m\omega\mathrm{a}}{(2n+1)\mathrm{R}}\cdot i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

To interpret these results, let Q_0 be the value of P_0 which would result from the magnetism inside the shell combined with the positive image (at the surface) of the magnetism outside the shell, then

$$\mathrm{Q}_0=\Sigma(\mathrm{A}+\mathrm{A}')\left(\frac{\mathrm{b}}{r}\right)^{n+1}e^{im\phi}\mathrm{P}_m{}^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and

$$\mathrm{P}=\Sigma\mathrm{C}\left(\frac{\mathrm{b}}{r}\right)^{n+1}e^{im\phi}\mathrm{P}_m{}^n:$$

both the foregoing results merge into the equation

$$r^{\frac{1}{2}}\frac{d}{dr}(\mathrm{P}r^{\frac{1}{2}})=\frac{2\pi\omega\mathrm{a}}{\mathrm{R}}\frac{d}{d\phi}(\mathrm{P}+\mathrm{Q}_0)$$

or putting

$$\mathfrak{P}=\mathrm{P}r^{\frac{1}{2}},\ \mathfrak{Q}=\mathrm{Q}_0r^{\frac{1}{2}},\ \varpi=\frac{2\pi\omega}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

$$r\frac{d\mathfrak{P}}{dr}-\varpi\mathrm{a}\frac{d\mathfrak{P}}{d\phi}=\varpi\mathrm{a}\frac{d\mathfrak{Q}}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

When $\mathrm{a}=\infty$, and $r=\mathrm{a}+z$, the above equation becomes

$$\frac{d\mathfrak{P}}{dz}-\varpi\frac{d\mathfrak{P}}{d\phi}=\varpi\frac{d\mathfrak{Q}}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

indicating the spiral trail of images obtained by MAXWELL.

If in the sphere we put $r=e^{\frac{\rho}{\mathrm{a}}}$, we have

$$\frac{d\mathfrak{P}}{d\rho}-\varpi\frac{d\mathfrak{P}}{d\phi}=\varpi\frac{d\mathfrak{Q}}{d\phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

which is analogous to MAXWELL'S equation, and may be interpreted in a similar manner.

§ 19. We shall briefly verify these results by a direct investigation of the case of an infinitely thin shell.

Putting as before

$$u=0,\ v=-\frac{d\Phi}{\sin\theta d\phi},\ w=\frac{d\Phi}{d\theta}\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (36)$$

and denoting the vector potential of the shell by

$$F=0,\ G=-\frac{dP}{\sin\theta d\phi},\ H=\frac{dP}{d\theta}\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (37)$$

we know (MAXWELL'S 'Electricity and Magnetism,' Vol. II., Art. 670) that P is the scalar potential of a shell of matter coinciding with the surface of the sphere where surface density is

$$-\frac{\Phi}{a}\ \ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (38)$$

The equations of the currents on the sheet are

$$\left.\begin{aligned}-\frac{R}{\sin\theta}\frac{d\Phi}{d\phi}&=\omega r\sin\theta.\alpha-\frac{d\psi}{rd\theta}\\ R\frac{d\Phi}{d\theta}&=\qquad\qquad\ -\frac{d\psi}{r\sin\theta d\phi}\end{aligned}\right\}\ \ .\ .\ .\ .\ .\ .\ .\ \ (39)$$

wherein, also,

$$\alpha=\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}.\sin\theta\frac{d\overline{P+P_0}}{d\theta}+\frac{1}{\sin^2\theta}\frac{d^2\overline{P+P_0}}{d\phi^2}\right]$$

as in the foregoing articles, and where also r is to be put equal to a.

All these equations are satisfied by

$$\Phi=-\frac{d\chi}{d\phi},\ \psi=Ra\sin\theta\frac{d\chi}{d\theta},\ R\chi=\omega(P+P_0)\ \ .\ .\ .\ .\ .\ .\ .\ \ (40)$$

ψ now referring only to the surface of the sheet.

$$\text{Outside the sheet}\quad P_1=C\left(\frac{a}{r}\right)^{n+1}e^{im\phi}P_m{}^n,$$

$$\text{Inside}\quad\text{,,}\quad P_2=C\left(\frac{r}{a}\right)^{n}e^{im\phi}P_m{}^n,$$

P_1 and P_2 being the values due to the currents in the sheet. At the sheet

$$\left(\frac{d\mathrm{P}_1}{dr}-\frac{d\mathrm{P}_2}{dr}\right)_{r=\mathrm{a}}=-4\pi\Phi$$

$$\therefore\ (2n+1)\frac{\mathrm{C}}{\mathrm{a}}=4\pi\mathrm{E}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (41)$$

where

$$\Phi=\Sigma(\mathrm{E}e^{im\phi}\mathrm{P}_m{}^n).$$

But P_0 being $=\Sigma\mathfrak{A}e^{im\phi}\mathrm{P}_m{}^n$ at the surface of the sheet, equations (40) give

$$\mathrm{RE}=-\omega mi(\mathrm{C}+\mathfrak{A}),$$

and therefore

$$\mathrm{C}=-\frac{4\pi\omega m\mathrm{a}}{2n+1}i.(\mathrm{C}+\mathfrak{A})\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ \ (42)$$

the same result as obtained in the previous article.